Plenary talks

Sam Hughes: Distinguishing free-by-cyclic groups by their finite quotient groups

Jean-Pierre Mutanguha: Canonical forms for free gp. automorphisms

Michelle Chu Arithmetic hyperb. mfds. and their finite covers

Rylee Lyman Groups acting on trees and their deformations

Stephan Stadler CAT(0)-spaces of higher rank

Ignacio Vergara Uniformly Lipschitz affine actions on subspaces of L^1

Annette Karrer From Stallings' theorem to connected components of Morse boundaries of graphs of groups

Distinguishing free-by-cyclic groups by their finite quotients

joint w. Monika Kudlinska

G a fin. gen. gp.

$\{G/H \mid H \trianglelefteq_f G\}/_{\cong} =: G_{Fin}$

$\hat{G} := \varprojlim_{H \trianglelefteq_f G} G/H$ $< \prod_{H \trianglelefteq_f G} G/H$

Thm. (Dixon-Formanek-Poland-Ribes '82)
Γ, Λ discrete fin. gen. gps. Then
$\hat{\Gamma} \cong \hat{\Lambda} \iff \Gamma_{Fin} = \Lambda_{Fin}$

Q: Given a collection G_{Fin}, when can we construct G? (Amongst fin. gen. res. fin. gps)

G is resid. finite iff $G \hookrightarrow \hat{G}$

Suppose for G the answer is "yes".
Then, G is profinitely rigid.

Ex.:
- fin. gen. Ab. gps. are profinitely rigid
- Bridson-McReynolds-Reid-Spitler, '19 :
  Some fin. vol. $\pi_1(M^3_{hyp.})$

everything is fin. gen. resid. fin.

If $\exists$ at most fin. many $G_1, \dots, G_k$ s.t. $\forall i,j$: $\hat{G}_i \cong \hat{G}_j$, then G_i is almost profinitely rigid

Ex.:
- Pitchell, '70s: almost rigidity of nilpotent gps.
- Baumslag, '74: $\exists G_1, G_2$:
  $1 \to \mathbb{Z}/25 \underset{\substack{\cdot 6\\ \cdot 11}}{\to} G_i \to \mathbb{Z} \to 1$

Open

$SL_n(\mathbb{Z})$ $n \geq 2$	Free gps.
$Out(F_n)$ $n \geq 2$	Surface gps.
$MCG(\Sigma_g)$ $\forall g, b$	

Liu '23: Fin. vol. hyperb. 3-mfds. are almost profin. rigid. amongst 3-mfd. gps.

Agol '12: also Wise
Every fin. vol. hyp. 3-mfd. is virtually fibred

A 3-mfd. is fibred (over the circle) if $\exists$ a fibre bundle $\Sigma_{g,b} \to M \to S^1$
on homotopy:

$$\begin{tikzcd}
\pi_2(S^1) \arrow[r] \arrow[d, equal] & \pi_1(\Sigma_{g,b}) \arrow[r] & \pi_1(M) \arrow[r] \arrow[d, equal] & \pi_1(S^1) \arrow[d, equal] & \\
1 \arrow[r] & \pi_1(\Sigma_{g,b}) \arrow[r] & \pi_1(M) \arrow[r] & \mathbb{Z} \arrow[r] & 1
\end{tikzcd}$$

Taibin-Zapirain '20:
If N is a closed conn. orientable 3-mfd. and $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$ s.t. M is fibred, then N is fibred.

Encode dynamics of $\mathbb{Z} \curvearrowright \Sigma_{g,b}$ into the homology
"Twisted Alexander polyn. and Reidemeister torsion"

Encode the stretch factor λ of the monodr. $\mathbb{Z} \curvearrowright \Sigma_{g,b}$

$\exists$ only fin. many fin. vol. hyp. 3-mfds. with fibre $\Sigma_{g,b}$ and stretch at most λ.

G is free-by-cyclic, if
$\exists$ a SES $1 \to F_n \to G \to \mathbb{Z} \to 1$
$\Rightarrow G = F_n \rtimes_\varphi \mathbb{Z}$

$\Phi \in Out(F_n)$, φ Lift in $Aut(F_n)$
A top. repr. for Φ is a pair (f, G) where
- G is a graph s.t. $\pi_1(G) \cong F_n$
- $f: G \to G$ is homot. equiv. s.t.
  - $f(G^{(0)}) \subseteq G^{(0)}$
  - f is locally inj. on the interior of edges
- f induces Φ on G.

The incidence matrix A of is the matrix (a_{ij}) where $a_{ij} = \#$ occurrences of the edge e_i in the path $f(e_j)$

A filtration of length l is a sequence
$\{\} = G_0 \subseteq G_1 \subseteq \dots \subseteq G_l = G$ s.t.
$f(G_i) \subseteq G_i$
Let $\Lambda_i := \overline{G_i - G_{i-1}}$

A filtration is maximal if the square submatrix A_i corr. to Λ_i is irreducible
$\hookrightarrow \nexists X$ a perm. matrix s.t. $XA_iX^{-1} = \begin{bmatrix} C & D \\ 0 & E \end{bmatrix}$

If (f, G) admits a maximal filtration of length 1, we say (f, G) is irreducible.

Φ is irreducible if every top. repr. of Φ with no leaves or inv. forests is irreducible.

[Perron-Frobenius]
If Φ is irreducible, then A is an irred. matrix
$\Rightarrow \exists$ a unique maximal eigenvalue λ_f
λ_f is the stretch factor

Lemma:
$n \geq 2$, $C \geq 1$. Then $\exists$ at most fin. many equiv. classes of top. reprs. (f, G) with $\pi_1(G) \cong F_n$ s.t.
- f is irr.
- $\lambda_f \leq C$

Thm. [Hughes-Kudlinska]
Let G be a generic free-by-cyclic group. Then G is almost profinitely rigid amongst all free-by-cyclic groups.

Canonical forms for free gp. Automorphisms

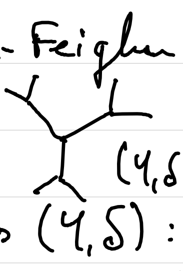
F fin. gen. free gp. $(\pi_1(\text{graph}))$
$\circlearrowleft$
$[\phi]$ outer autom. ~ induced by autom. g of graph Γ

Def.:
$[\phi]$ has fin. order if $\exists n \geq 1: g^n \simeq id$
$[\phi]$ is reducible: "$\exists$ proper ess. subgraph that g maps to itself"

I Irred. Automorphisms

$\phi: F \to F$ inf. order, irred.

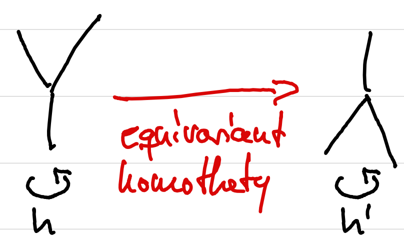
Thm. (Bestvina-Feighn-Handel, '96)

① $\exists F \overset{\text{isom.}}{\curvearrowright}$  (Y,δ) : • minimal • trivial arc stab.

② $h: (Y,\delta) \to (Y,\delta)$: • ϕ-equivariant: $\forall p \in Y: \forall a \in F: h(a \cdot p) = \phi(a) \cdot h(p)$
• expanding homothety: $\exists \lambda > 1: \forall p,q \in Y: \delta(h(p),h(q)) = \lambda\delta(p,q)$

③ $\forall a \in F$: a is Y-loxodromic $:\Longleftrightarrow$ $[a]$ grows exp. w.r.t. $[\phi]$

(✳) the triple (Y,δ,h) is universal!!! w.r.t. 1-3



"Jordan-canonical form"

Pf. (Sketch, existence part)
(Γ,d) metric graph.
$\tilde{g}: (\tilde{\Gamma},d) \to (\tilde{\Gamma},d)$ ϕ-equiv. (lift to univ. cover)
$\Rightarrow \tilde{g}$ is K-Lipschitz and inf. order $\Rightarrow K > 1$ (Exc.)

$$\forall p,q \in \tilde{\Gamma}: \lim_{n\to\infty} \frac{d(\tilde{g}^n(p),\tilde{g}^n(q))}{K^n} =: \delta(p,q)$$

Set $(Y,\delta) := (\tilde{\Gamma},\delta)/_{\sim_\delta}$ is an $\mathbb{R}$-tree
$\circlearrowleft$ h ϕ-equiv. and K-homothety
ind. by g

In general, $\delta \equiv 0$ is possible

[Bestvina-Handel, '92]
$[\phi]$ irred. $\Rightarrow$ inf. K-Lipschitz constant is realised $\Rightarrow$ ③

:)

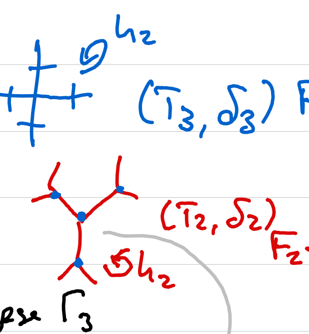
II Arbitrary Automorphisms (by example)

$F = F(a,b,c,d,e,f)$

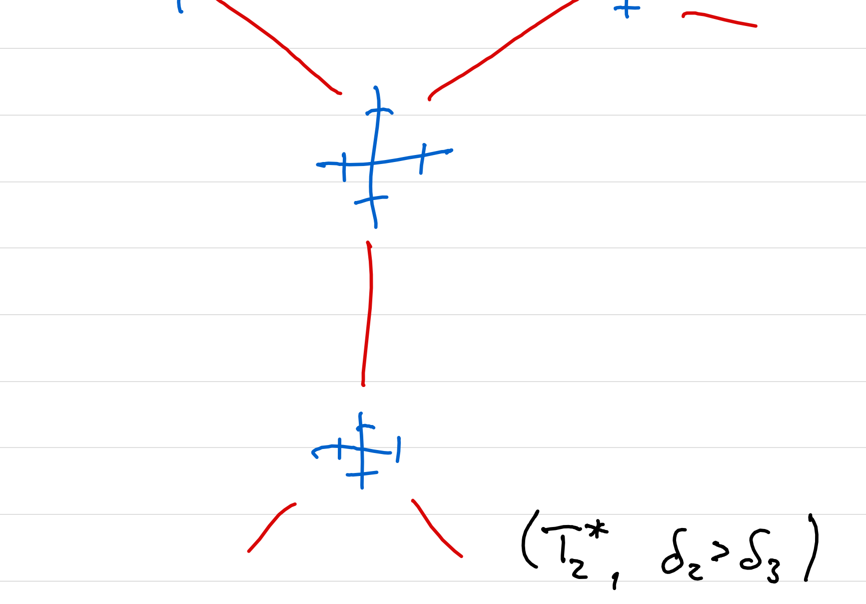
φ: $a \longmapsto ab$
$b \longmapsto bab$ } $F_3 = \langle a,b \rangle$
$c \longmapsto [a,b]cd$
$d \longmapsto dcd$ } $F_2 = \langle a,b,c,d \rangle$
$e \longmapsto cef$
$f \longmapsto fef$

Let $\Gamma =$ , $\tau: \Gamma \to \Gamma$ "obvious"

Pf. (Sketch)
$\Gamma_3 \rightsquigarrow \tilde{\Gamma} \circlearrowleft \tilde{\tau}_3$ $\overset{\text{degeneration}}{\rightsquigarrow}$ (T_3,δ_3) F_3-action
$(\Gamma_2,\Gamma_3) \rightsquigarrow$ "relative univ. covers" $\overset{\text{degeneration}}{\rightsquigarrow}$ (T_2,δ_2) F_2-action
– want to collapse Γ_3



blue pts are actually dense



$(T_2^*, \delta_2 > \delta_3)$

III Attracting Laminations

$\ldots aaaa \ldots \overset{\phi}{\longmapsto} \ldots (ab) \ldots \overset{\phi}{\longmapsto} \ldots (abbab) \ldots$
$\ldots$ {some bi-inf. words in a's and b's} – Λ_3

$(c) \overset{\phi}{\longmapsto} ([a,b]cd) \overset{\phi}{\longmapsto} ([a,b]^2cddcd) \longmapsto \ldots$
$\ldots \longmapsto$ {some bi-inf. words in $[a,b], c, d$} – Λ_2

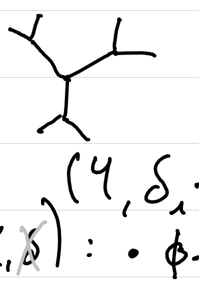
$(e) \overset{\phi}{\longmapsto} \underset{- - - - - -}{[ab]cdcefef} \longmapsto$ {some bi-inf. words in $[a,b], c, d, e, f$} – Λ_1

Poset: Λ_1 $\Lambda_3 \leftarrow$ topmost
|
Λ_2

IV Automorphisms

$\phi: F \to F \rightsquigarrow$ topmost $\{\Lambda_1, \ldots, \Lambda_k\}$

Thm. (M.)

① $\exists F \overset{\text{isom.}}{\curvearrowright}$  $(Y, \delta_1 + \ldots + \delta_k)$: • minimal • trivial arc stab.

② $h: (Y,\delta) \to (Y,\delta)$: • ϕ-equivariant: $\forall p \in Y: \forall a \in F: h(a \cdot p) = \phi(a) \cdot h(p)$
• expanding ~~homothety~~: dilation
$\exists \lambda > 1: \forall p,q \in Y: \delta(h(p),h(q)) = \lambda\delta(p,q)$

③ $\forall a \in F$: a is Y-loxodromic $:\Longleftrightarrow$ $[\phi^n(a)]$ limits to Λ_i

(✳) the triple (Y,δ,h) is universal!!! w.r.t. 1-3

Arithmetic hyperbolic manifolds and their finite covers

(arithmetic subgps. of $\text{Isom}(\mathbb{H}^n)$ and their finite index subgps.)

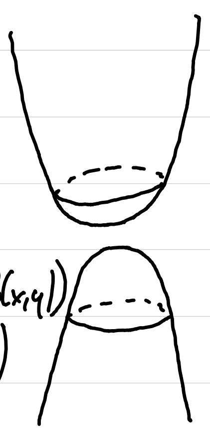
[hyperbolic: real hyperbolic]

1. $\mathbb{H}^n$

Let $Q = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{matrix} \}n \\ \}1 \end{matrix}$

$x \in \mathbb{R}^{n+1}$, $q(x) = x^T Q x$

$\{q(x) = -1\} \to$ hyperboloid



metric $d(x,y) = \operatorname{arccosh}(-B(x,y))$

$B(x,y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$

Isom. of this:

$$O(Q;\mathbb{R}) := \{A \in GL(n+1,\mathbb{R}) \mid AQA^T = Q\}$$

$\to$ identify $\mathbb{H}^n$ with upper sheet

$\text{Isom}(\mathbb{H}^n) = O^+(Q;\mathbb{R})$

$\mathbb{H}^n = SO(n)\backslash SO^+(Q;\mathbb{R})$

$\hookleftarrow$ stab. of pt.

If $\Gamma \overset{\text{disc.}}{\leq} \text{Isom}(\mathbb{H}^n)$, then $\underbrace{\Gamma\backslash\mathbb{H}^n}_{=:M}$ is a hyperb. orbif.

If M has finite volume, then we say that Γ is a lattice in $\text{Isom}(\mathbb{H}^n)$

Mostow-Prasad rigidity

$M_1 = \Gamma_1\backslash\mathbb{H}^n$, $M_2 = \Gamma_2\backslash\mathbb{H}^n$ fin. vol. $(n > 2)$

M_1, M_2 are isometric

$\Updownarrow$

Γ_1, Γ_2 are isomorphic

2. How to construct lattices: arithm. subgps.

Start with k a totally real number field

$(\forall \sigma: k \to \mathbb{C}$ lands in $\mathbb{R})$ e.g. $\mathbb{Q}$

Let R_k be the ring of integers in k

Let Q be a symm. matrix over k of signature $(n,1)$ s.t. $\forall$ other σ: Q^σ has sign. $(n+1,0)$

e.g. $\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 2 & 1 \\ & & 1 & 2 \end{pmatrix}$

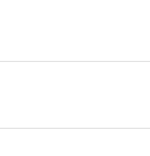
Borel-Harish-Chandra

If k, Q are as above, then $\Gamma := O^+(Q;R_k)$ is a lattice in $O^+(Q;\mathbb{R}) \cong \text{Isom}(\mathbb{H}^n)$

Def.: A lattice $\Gamma \leq \mathbb{H}^n$ is arithmetic (of simplest type) if it is commensurable w/ one as above

Γ, Γ' comm. if $\Gamma \cap \Gamma'$ has fin. index in Γ and in Γ'

corr. lattice $(\cong \pi_1(S^3 \setminus \text{thickened } 4_1)) \leftarrow$ e.g.



has index 12 in $O^+(Q,\mathbb{Z})$ from ex.

These come with a natural family of fin. covers:

If $\mathcal{J}$ is an ideal in R_k, then e.g. (n) in $\mathbb{Z}$

$$\begin{tikzcd} \pi_{\mathcal{J}}: \Gamma \arrow[r] \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}"] & GL_{n+1}(R_k/\mathcal{J}) \\ GL_{n+1}(R_k) & \end{tikzcd}$$

$\Gamma(\mathcal{J}) = \ker(\pi_{\mathcal{J}})$

$\to$ fin. index p congruence subgp. of level $\mathcal{J}$

$\hat{\Gamma} \longrightarrow \overline{\Gamma}$

Say that a lattice Γ in G has congruence subgp. property (CSP) if ker is finite

(BIG) Conj.: lattice Γ in G has CSP iff $\operatorname{rank}(G) > 1$

alt. def. / big thm. (Bader-Fisher-Miller-Stover)

If $M \supseteq$ at least 1 tot. geod. codim-1 submfd., then M is arithmetic iff it contains ∞-many)) (immersed)

3. Virtual properties

(pre Agol, Wise)

Bergeron-Haglund-Wise

Γ arithm. simplest type in $\text{Isom}(\mathbb{H}^n)$ have a congruence subgp.

$\hookrightarrow$ RACG

Chu

If Γ is defined over $\mathbb{Q}$ and $n \leq 6^*$, then $\Gamma(2) \hookrightarrow$ RACG

residual finiteness

separability (RF)

Deblois-Miller-Patel in some $n=3$, bounds on deg. of cover lin. w.r.t. length of $\gamma \in \Gamma$

Growth in towers of covers

- Heegaard genus gradient:

  $M^3 = H_g \cup_{\Sigma_g} H_g$

  $g_H(M) = \min g$

  Lackenby: for congruence towers:

  $g_H(M_i)$ grows lin. w.r.t. degree
- Trisections: $g_T(M^4)$ lin. w.r.t. degree (Chu-Tillmann)

RFRS (res. fin. resid. solv.)

Agol-Stover: for some Bianchi gps.

$\Gamma \geq \Gamma(2) \geq \Gamma(2^3) \geq \dots$ is RFRS

Growth in homology

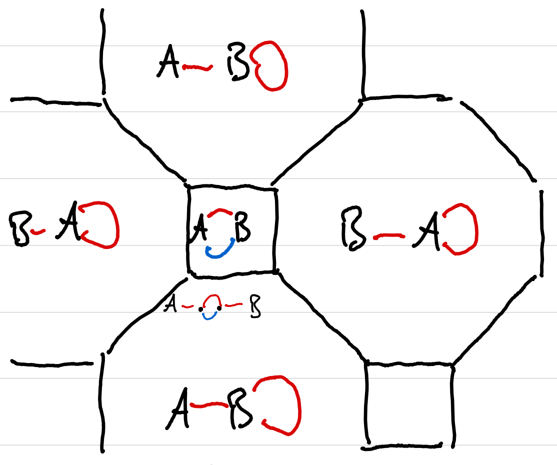
Lück: $\lim\limits_{i\to\infty} \frac{b_1(M_i)}{\text{degree}(M_i \to M)}$

Conj.: $M^3 = \Gamma\backslash\mathbb{H}^3$

$\exists$ regular tower of covers:

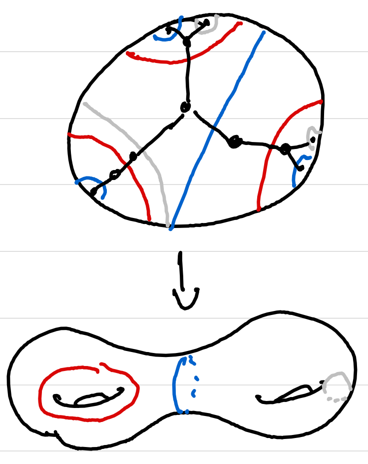
$$\lim_{i\to\infty} \frac{\log(|\text{Tor}(H_1(M_i))|)}{\text{degree}(M \to M_i)} = \frac{\text{vol}(M)}{6\pi}$$

Groups acting on trees and their deformations



$\pi_1(\Gamma) \circlearrowright \Gamma$

$\forall \Gamma$ conn., $\chi(\Gamma) = 1-n$, $F_n \circlearrowright \tilde{\Gamma}$



Ex.:

$\underset{\text{gp.}}{G} \circlearrowright \underset{\text{tree}}{T}$ $\quad \Phi : G \to G$ automorphism

get $(g,p) \in G \times T$

$\downarrow$

$\Phi(g) \cdot p$

Exc.:

If $\Phi \in \mathrm{Inn}(G)$, $\exists$ G-equiv. homeom. $(T, \cdot) \longrightarrow (T, \cdot_{\Phi})$

$\exists\ \mathrm{Aut}(G) \underset{\text{on the right}}{\circlearrowright} \{T \mid G \circlearrowright T\} / \sim_{\substack{G\text{-equiv}\\ \text{homeo}}}$

could also consider isom. → larger sp.

get $\mathrm{Out}(G)$- action

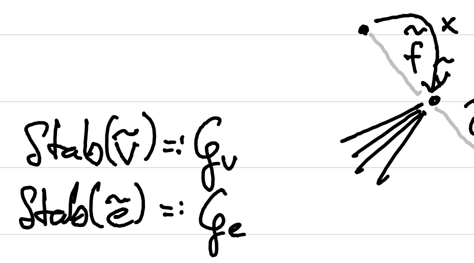
Graphs of groups

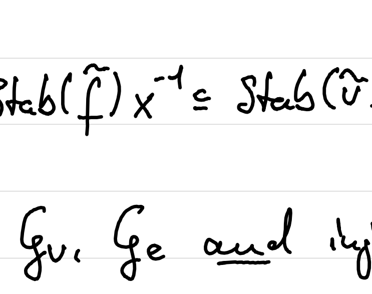
$G \circlearrowright T$ $\quad G \backslash\!\backslash T$ is already a graph

(unless $\exists g$: $\tilde{e} \xrightarrow{g} \bar{\tilde{e}}$)

inversion in edges

$G \backslash\!\backslash T$





$v \longmapsto \mathrm{Stab}(\tilde{v}) =: \mathcal{G}_v$

$e \longmapsto \mathrm{Stab}(\tilde{e}) =: \mathcal{G}_e$

$\mathcal{G}_e \begin{smallmatrix} \le \mathcal{G}_v \\ \le \mathcal{G}_w \end{smallmatrix}$

$x\, \mathrm{Stab}(\tilde{f})\, x^{-1} \le \mathrm{Stab}(\tilde{v})$

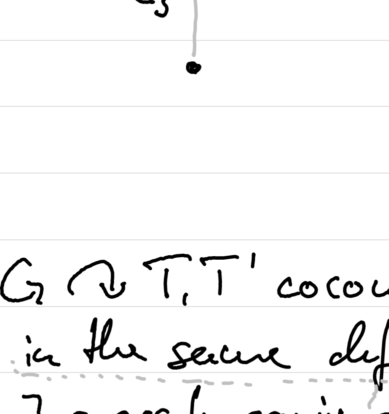
$G \backslash\!\backslash T$ is a graph w/ $\mathcal{G}_v$, $\mathcal{G}_e$ and inj. homom.

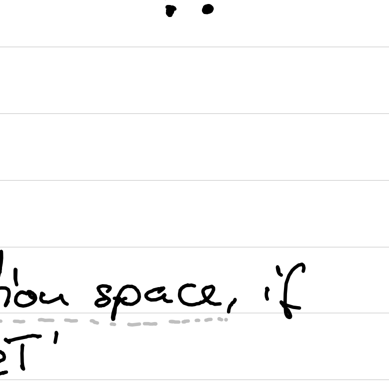
$$\begin{tikzcd} & \mathcal{G}_v \\ \mathcal{G}_e \arrow[ur] \arrow[dr] & \\ & \mathcal{G}_w \end{tikzcd}$$

Thm.: (Bass-Serre, '77)

$\forall \mathcal{G}$: $\exists\ G =: \pi_1(\mathcal{G})$ and $T = \tilde{\mathcal{G}}$ s.t. $G \backslash\!\backslash T = \mathcal{G}$

Moreover, $\exists$ simple presentation for G based on $\mathcal{G}$





$G \circlearrowright T, T'$ cocompact

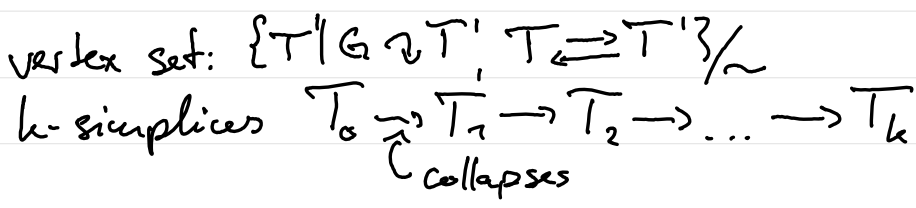
in the same deformation space, if

$\exists$ coarsely equiv. q-i $T \rightleftarrows T'$

Thm. (Forester, Guirardel-Levitt)

↑ is equivalent to:

- $\exists$ G-equiv. cts maps $T \rightleftarrows T'$
- $H \le G$ is elliptic in $T \Leftrightarrow$ in T'
- (Lyman) $G \backslash\!\backslash T \underset{\substack{\text{homot.}\\ \text{equiv.}}}{\xrightarrow{\simeq}} G \backslash\!\backslash T'$
- "collapse-expand path" from T to T'



vertex set: $\{T' \mid G \circlearrowright T',\ T \rightleftarrows T'\} / \sim$

k-simplices $T_0 \to T_1 \to T_2 \to \dots \to T_k$

collapses

"<u>spine</u> of def. space" $\mathcal{K} = \mathcal{K}(T)$

Forester: $\mathcal{K}$ is connected

G-L: $\mathcal{K}$ is contractible

$G = F_n$

$G \circlearrowright T$ freely

$K(T)$ is spine of

Outerspace CV_n

"Mod(T)"

"MCG($\mathcal{G}$)" $\le \mathrm{Out}(G)$ s.t.

$\forall H$ elliptic, Φ rep. φ

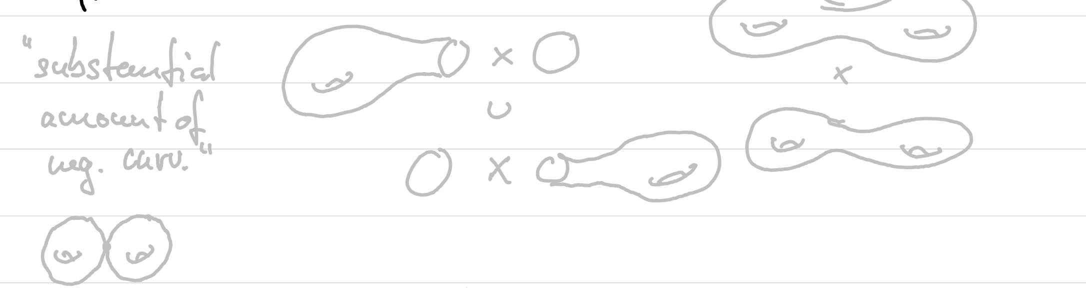
$\Phi(H)$ is elliptic in T

CAT(0)-spaces of higher rank

General setting: X loc. cpct. geod. complete CAT(0) space with $\Gamma \curvearrowright X$ geom.

3 "boxes"

almost hyperbolic   collapsible   uniflat



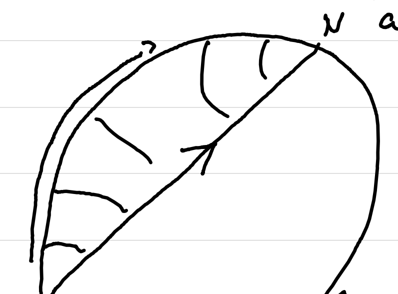
boxes not disj., not always clear how to put stuff

2D-mfd.s

Σ_g, $g \geq 2$ T^2

3D

hyp. piece   graph mfd. $\Sigma \times S^1$, T^3



v "axial isom." on CAT(0)-space

$\mathrm{rk}(c) = 1 \Leftrightarrow$ no flat half-plane

$\mathrm{rk}(X) = 1 \Leftrightarrow$ contains a rank 1 geodesic

Conjecture A (Closing Lemma)
If X has rank 1, then X contains a Γ-periodic rank 1 geodesic.

Conjecture B (Diameter rigidity)
If X has higher rank, then it is rigid, i.e. a Riemannian symm. space, Euklidean building, or a product.

Known Cases: (A+B)

- Hadamard mfds (Ballmann, '78(A), '85 (B)) (Bu-Sp, '87) (Eb-He, '90)
- Cube complexes (Ca-Sa, '11)
- Euklidean complex of dim ≤ 3 (Ba-Br, '95, '00)
- + other related things

Thm. 1:
Conj. A and B hold if X does not contain a 3-flat.
Cor. All such gps. Γ satisfy Tits Alternative

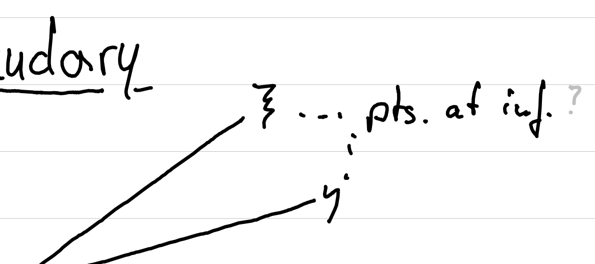
Thm. 2:
Let X be a loc. cpct. CAT(0)-space with $\Gamma \curvearrowright X$ geom. Suppose every geodesic lies in an n-flat, $n \geq 2$. If X contains a periodic Morse n-flat, then X is rigid (-> conj. B)

Rmk.

- a flat F is Morse $:\Leftrightarrow$ no flat half-space H $\underset{\text{HKS}}{\Leftrightarrow}$ Morse (quasi-)flat

- $\Gamma \curvearrowright X_{\text{model}}$ Γ-periodic n-flats are dense

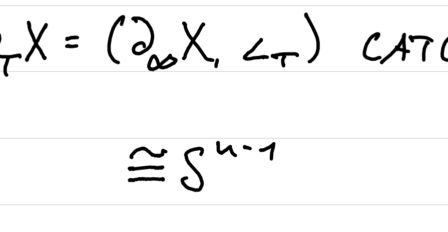
Tits boundary



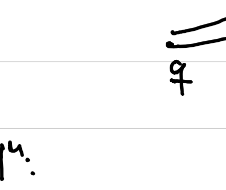
$$\angle_T(\xi, \eta) = \sup_{p \in X} \angle_p(\xi, \eta)$$

Fact: $\partial_T X = (\partial_\infty X, \angle_T)$ CAT(1)

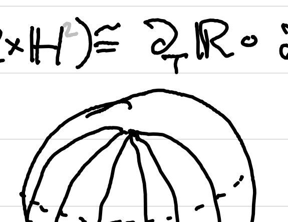
$\partial_T \mathbb{R}^n$: $\cong S^{n-1}$



$\partial_T \mathbb{H}^n$:   = discrete space



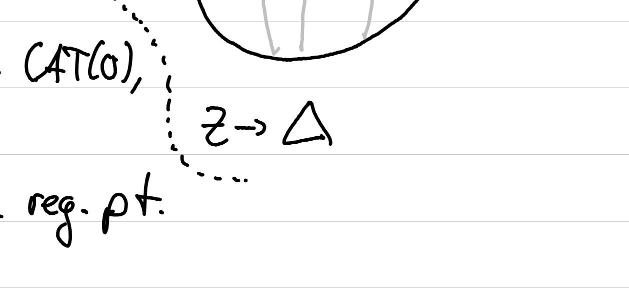
$\partial_T(\mathbb{R} \times \mathbb{H}^2) \cong \partial_T \mathbb{R} \circ \partial_T \mathbb{H}$



$\partial_T {}^{SL(3)}/_{SO(3)}$ $\mathrm{Flag}(\mathbb{R}^3)$

Thm. (Lytchak '05)
Z CAT(1) geod. complete,
$A \subset Z$ proper closed, contains with every pt. all its antipodes, then Z is $Z = Z_1 \circ Z_2$ or spherical building

Conj. (Lytchak)
Regular pt. conj.:
Let X be a loc. cpct. CAT(0),
$\Gamma \curvearrowright X$ geom.,
then $\partial_T X$ contains a reg. pt.



Thm.
Periodic Morse flat ⇒ reg. pt.

Uniformly Lipschitz affine actions on subspaces of L^1

I Setting

Γ a countable gp.

E Banach space (e.g. $E = L^p(\Omega,\mu)$)

affine action: $\Gamma \overset{\sigma}{\circlearrowright} E$:

- $\pi: \Gamma \to \underbrace{GL(E)}_{\text{inv. lin. operators}}$ representation (group homom.)
- $b: \Gamma \to E$ "cocycle"

  $b(st) = \pi(s)b(t) + b(s)$

$\rightsquigarrow \sigma(s)\cdot v = \pi(s)v + b(s)$

"the only option" for isometric actions

__Particular case:__

$\Gamma \overset{\sigma}{\circlearrowright} E$ isometric

$[\forall s\in\Gamma: \forall v,w\in E: \|\sigma(s)v - \sigma(s)w\| = \|v-w\|]$

$$\Updownarrow$$

$[\forall s\in\Gamma: \forall v\in E: \|\pi(s)v\| = \|v\|]$

__Rmk.:__

- σ is proper iff b is proper
- σ has bounded orbits $\Leftrightarrow$ b is bounded

II Actions on L^2 (Hilbert spaces)

__Def.__ Γ has the __Haagerup property__ if it has a proper isometric action on a Hilbert space.

σ is proper if

$\forall v_0\in E: \forall M>0: \{s\in\Gamma \mid \|\sigma(s)v_0 - v_0\| \leq M\}$ is finite

__Examples:__

amenable gps., $\Gamma \curvearrowright X$ properly with X a CAT(0) cube complex, Baumslag-Solitar gps.

__Def.:__

Γ has Property (T) if __every__ isometric action on a Hilbert space has a fixed pt.

__Examples:__

Lattices in $Sp(n,1)$ $(n\geq 2)$, higher rank lattices, "most" hyperbolic gps.

__Rmk.:__

Prop. (T) + Haagerup prop. $\Leftrightarrow$ finite

Generalisation: Uniformly Lipschitz actions

$\exists C\geq 1: \forall s\in\Gamma: \forall v,w\in E: \|\sigma(s)v - \sigma(s)w\| \leq C\|v-w\|$

$$\Updownarrow$$

$\exists C\geq 1: \forall s\in\Gamma: \forall v\in E: \|\pi(s)v\| \leq C\|v\|$ $\rightarrow$ unif. bdd. representation

__Thm.:__ (Nishikawa '20)

Lattices in $Sp(n,1)$ have proper unif. Lipschitz affine actions on L^2

__Conj.__ (Shalom):

$\uparrow$ true for all hyp. gps.

III Actions on $E \subset L^1$

Why subspaces of L^1?

__Thm.__ (Chatterji-Druţu-Haglund, '07)

Let Γ be a countable gp.

a) If Γ has the Haagerup prop., then it has a proper isometric action on L^p for any $p\in[1,\infty)$

b) If Γ has a proper isometric action on a subset of L^p (for some $p\in[1,2]$), then it has the Haagerup prop.

__Corollary__

Haagerup prop. $\Leftrightarrow$ proper isom. action on $E \subseteq L^1$

__Def.:__

Let X be a set. $\chi: X\times X \to [0,\infty]$ is called a CND kernel if:

(conditionally negative definite)

- $\forall x,y\in X: \chi(x,y) = \chi(y,x)$
- $\forall x\in X: \chi(x,x) = 0$
- $\forall n\geq 1: \forall x_1,\dots,x_n\in X: \forall \lambda_1,\dots,\lambda_n\in\mathbb{R}, \sum_i \lambda_i = 0:$

  $\sum_{i,j=1}^{n} \lambda_i\lambda_j\,\chi(x_i,x_j) \leq 0$

__Thm.__ (V. 2022)

Let Γ be a countable gp., $\chi: X\times X \to [0,\infty)$ a CND kernel, $C>0$ s.t.

$\forall s,x,y\in\Gamma: \chi(sx,sy) \leq \chi(x,y) + C$

Then $\exists$ (Ω,μ) meas. space, $E \subset L^1(\Omega,\mu)$ subspace, $\pi: \Gamma\to GL(E)$ a unif. bdd. repr., $b:\Gamma\to E$ a cocycle s.t.

$$\forall s\in\Gamma\setminus\{e\}: \|b(s)\|_E = \chi(s,e)^{1/2} + 2$$

__Applications:__

1) $\Gamma \circlearrowright X_1 + \dots + X_n$, X_i quasi-tree

$\left.\begin{array}{l}\text{- MCG}\\ \text{- res. fin. hyp. gps.}\end{array}\right\}$ Bestvina-Bromberg-Fujiwara — proper actions

Acylindrical hyperb. gps. — unbounded orbits

2) hyperbolic gps. — proper actions

From Stallings' theorem to connected components of Morse boundaries of graphs of groups

j. with Elia Fioravanti

I Stallings' thm
II Boundaries
III Thm.

I Let $\Gamma = \langle S \rangle$, $|S| < \infty$

__Def:__

$$\text{ends}(\text{Cay}(\Gamma, S)) := \{\gamma : [0,\infty) \to \text{Cay}(\Gamma,S) \text{ geod. rays}, \gamma(0)=id\} / \sim$$

$\gamma_1 \sim \gamma_2$ iff $\forall R>0$: γ_1, γ_2 end in the same component of $\text{Cay}(\Gamma,S) \setminus B_R(id)$

$\text{ends}(\Gamma) := \text{ends}(\text{Cay}(\Gamma,S))$ Fact: well-defined

[photo]

__Thm.__ (Stallings '68, Hopf '43)

$|\text{ends}(\Gamma)|$

$0 \Leftrightarrow |\Gamma| < \infty$

$2 \Leftrightarrow [\Gamma : \mathbb{Z}]$

$\infty \Leftrightarrow \Gamma$ splits non-trivially along C, $|C| < \infty$

1

Amalgamated free products

$A = \langle S_A | R_A \rangle$
$B = \langle S_B | R_B \rangle$
$C = \langle S_c \rangle$

$$\begin{tikzcd} & A \arrow[dr] & \\ C \arrow[ur, "\Phi_A"] \arrow[dr, "\Phi_B"'] & \circlearrowleft & A *_c B \\ & B \arrow[ur] & \end{tikzcd}$$

$A *_c B = \langle S_A \cup S_B | R_A \cup R_B \cup \{\Phi_A(c) = \Phi_B(c)\} \rangle$

__Ex.:__ (Seifert-van Kampen)

__Def:__
Γ splits along a group C iff
$\Gamma = A *_c B$ or $\Gamma = A *_c$ HNN extension
non-triv.: $[A:C] \geq 3$
$[B:C] \geq 2$

II

$$\partial\, \text{Cay}(\Gamma,S) := \left\{ \begin{matrix} \gamma : [0,\infty) \to \text{Cay}(\Gamma,S) \\ \text{geod. rays}, \gamma(0) = id \end{matrix} \right\} / \sim$$

$\gamma_1 \sim \gamma_2$ iff $\exists\, C$ s.t. $\text{im}(\gamma_1) \subseteq N_C(\text{im}(\gamma_2))$ __and__ $\text{im}(\gamma_2) \subseteq N_C(\text{im}(\gamma_1))$

topology: fellow-topology

__Thm.:__ (Gromov)
If Γ is hyperbolic, $\Gamma = \langle S \rangle$
$\partial\Gamma := (\partial\, \text{Cay}(\Gamma,S), \text{fellow-travel-top.})$
is well-def., metrisable called __Gromov bdary.__

__Q:__ $\exists$ variant of Stallings' thm. for $\overset{\text{hyperbolic}}{\partial\Gamma}$?

__Thm.__ (Gromov)
Γ hyp. $\partial\Gamma$

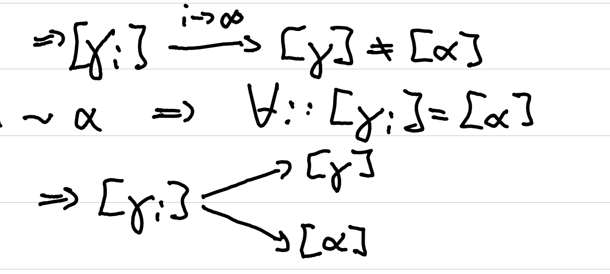
$\{\} \Leftrightarrow |\Gamma| < \infty$

2 pts. $\Leftrightarrow [\Gamma : \mathbb{Z}] < \infty$

∞ components $\Leftrightarrow \Gamma$ splits non-triv. along C, $|C| < \infty$

connected + non-empty $\Leftrightarrow$

__Q:__ What if Γ is not hyperbolic?
Problem: Limit of sequences might not be unique
e.g. $\mathbb{Z}^2$:



$$\gamma_i := \beta_i \circ \alpha|_{[i,\infty)} \longrightarrow \gamma$$

$$\Rightarrow [\gamma_i] \xrightarrow{i \to \infty} [\gamma] \neq [\alpha]$$

$\forall i$: $\gamma_i \sim \alpha \Rightarrow \forall i$: $[\gamma_i] = [\alpha]$

$$\Rightarrow [\gamma_i] \begin{cases} \to [\gamma] \\ \to [\alpha] \end{cases}$$

III Morse boundary

__Def:__ (Morse geodesic ray α in $\text{Cay}(\Gamma,S)$)
α is __Morse__ if
for each $\left\{ \begin{matrix} K \geq 1 \\ L \geq 0 \end{matrix} \right\}$ $\exists\, D(K,L)$ s.t.:

$$\left(\begin{matrix} \beta \text{ is a } (K,L)\text{-q-geod.} \\ \text{w/ endpts. on } \alpha \end{matrix} \right) \Rightarrow \beta \in N_D(\alpha)$$

__Ex.:__ No ray in $\mathbb{Z}^2$ is Morse

Morse lemma: Γ hyp. $\Rightarrow$ every geod. ray is Morse

__Morse boundary:__ ∂_M (Charney-Sultan, Cordes)
$\Gamma = \langle S \rangle$, $|S| < \infty$

$$\partial_M \Gamma := \partial_M\, \text{Cay}(\Gamma,S) = \left\{ \begin{matrix} \gamma : [0,\infty) \to \text{Cay}(\Gamma,S) \\ \text{Morse}, \gamma(0) = id \end{matrix} \right\} / \sim \; + \text{ suitable topology}$$

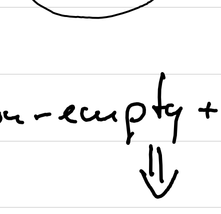
__Q:__ $\exists$ variant of Stallings' thm. for $\partial_M \Gamma$?

__Q:__ $\partial_M A *_c B$ where $\partial_M C = \{\}$?

__Thm.:__ (Fioravanti-Karrer)
If $\Gamma = \langle S \rangle$, $|S| < \infty$, $\Gamma = A *_c B$, C fin. gen.,
$C \hookrightarrow \text{Cay}(\Gamma,S)$ q-i emb., $\partial_M C = \{\}$
then for each conn. comp. K of $\partial_M \Gamma$: either
1) $|K| = 1$
2) K induced by $\partial_M A$
3) K induced by $\partial_M B$

__Rem.:__ Stronger variant for graphs of gps. (F-K)
S. Zbinden: $C = \{id\}$

__Application__

1) $\partial_M(\pi_1($  $))$ is totally disconnected

2) $\partial_M \Gamma$ non-empty + connected

$\Downarrow$

Γ splits as in thm.