Minicourse: Sofic groups, almost homom. and stability

G group, k field

$kG := \{ \sum a_\gamma \gamma \mid a_\gamma \in k, \gamma \in G \}$
unital ring ext. gp. mult.

G finite: $\mathbb{C}G = \bigoplus_{i=1}^{l} M_{n_i}\mathbb{C}$

l: no of. conj. classes $=$ # irred. reprs.

$G = \mathbb{Z}$
$\mathbb{C}\mathbb{Z} = \mathbb{C}[z, z^{-1}]$
no zero divisors
no nilpotent elements

Obs. If $a, b \in \mathbb{C}G$ (G finite or $\mathbb{Z}$),
then $ab = 1_{\mathbb{C}G} \Rightarrow ba = 1_{\mathbb{C}G}$

inj. self-maps are surj.

Def.:
A unital ring R is called finite if
$\forall a, b \in R$: $ab = 1 \Rightarrow ba = 1$

$\mathbb{C}\langle x, y \rangle / (xy - 1)$ is not finite

Kaplansky's conjecture: Group rings are finite.

Thm.:
If $\mathrm{char}(k) = 0$, then kG is finite.

Pf.:
WLOG: $k = \mathbb{C}$, $a, b \in \mathbb{C}G$, $ab = 1$
$\lambda: G \to U(\ell^2 G)$, $\lambda(h)\delta_g = \delta_{hg}$
$\ell^2 G = \overline{\mathrm{span}}(\delta_g, g \in G)$
$\rightsquigarrow \lambda: \mathbb{C}G \to B(\ell^2 G)$
$\lambda(\sum a_g g) := \sum a_g \lambda(g)$

$C^*_{red} G = \overline{\lambda(\mathbb{C}G)}^{\|\cdot\|} \subseteq B(\ell^2 G)$

$\tau: C^*_{red} G \to \mathbb{C}$, $\tau(a) := \langle a\delta_e, \delta_e \rangle$
$\tau(\lambda(\sum a_g g)) = a_e$

$a \geq 0$
$1 + a$ is inv.

Fact 1: 1) $\tau(ab) = \tau(ba)$, $\tau(1) = 1$
2) $a > 0$ ($:\Leftrightarrow a = b^* b$ in $C^*_{red} G$) $\Rightarrow \tau(a) > 0$
3) Every idempotent ($:\Leftrightarrow f^2 = f$) is conj. in $C^*_{red} G$ to a pos. idempotent
$\lambda(\sum a_g g)^* = \lambda(\underbrace{\sum \bar{a}_g g^{-1}}_{(\sum a_g g)^*})$

[$a \geq 0 \Leftrightarrow a = b b^* \Leftrightarrow \forall \xi \in \ell^2 G \;\; \langle a\xi, \xi \rangle \geq 0$]

Obs.:
$baba = ba \rightsquigarrow$ idempotent
$\tau(ba) = \tau(ab) = \tau(1) = 1$

$\rightsquigarrow 1 - ba$ is also an idempotent, and
$\tau(1 - ba) = 0$
$\rightsquigarrow f' := u(1 - ba)u^{-1}$ is a pos. idemp. of trace 0
$\rightsquigarrow f' = 0 \Rightarrow ba = 1$

☺

Open problem: Is $\mathbb{F}_2 G$ finite?

Known: If G finite, G Abelian

<u>Def.:</u>

G is <u>residually finite</u> $:\Longleftrightarrow$

$\forall F \subseteq G$ finite : $\exists$ homom. $\varphi: G \to H$ finite : $\varphi|_F$ inj.

<u>Thm.:</u>

G res. finite, then kG finite

<u>Def.:</u>

G is <u>amenable</u> if

$\forall \varepsilon > 0 : \forall S \subset G$ finite : $\exists F \subseteq G$ finite : $|SF| < (1+\varepsilon)|F|$

<u>Example:</u>

1) $S = \{-1, 0, 1\} \subseteq (\mathbb{Z}, +)$ $\quad \varepsilon = 10^{-17}$

$F = \{0, \dots, n\}$ $\quad$ for n large enough.

2) $\langle a, b | \rangle$ not amenable

$$S = supp(a) \cup supp(b)$$

$$(supp(\sum a_g g) = \{g \in G \mid a_g \neq 0\}$$

$$\begin{tikzcd} k[F] \arrow[r, "a"] \arrow[rr, bend right, "a_F"] & k[S \cdot F] \arrow[r, "\pi_F"] & k[F] \end{tikzcd}$$

$$a_F \, b_F \in End(k[F])$$

$$\simeq 1_{k[F]} \quad (:\Longleftrightarrow rk(a_F b_F - 1) \leq \varepsilon |F|$$

$$\Rightarrow rk(b_F a_F - 1) \leq 2\varepsilon |F|$$

[...]

Conclude that $ba = 1$

(2)

Sofic groups

Def.:

Let G be a gp. G is called sofic $:\iff$
$\forall F \subseteq G$ finite: $\forall \varepsilon > 0$: $\exists k \in \mathbb{N}$: $\exists \varphi: G \to \mathrm{Sym}(k)$ map:
1) whenever $g, h, gh \in F$, $\varphi(g)\varphi(h) \sim_\varepsilon \varphi(gh)$
2) $\forall g \in F \setminus \{e\}$: $\varphi(g)$ far from $\varphi(e)$

What does $\sim_\varepsilon$ mean?
$\exists$ normalised Hamming distance on $\mathrm{Sym}(k)$
$d(\sigma,\tau) := |\{i \mid 1 \le i \le k,\ \sigma(i) \neq \tau(i)\}| \cdot k^{-1} \in [0,1]$
$\sigma \sim_\varepsilon \tau \ :\iff d(\sigma,\tau) \le \varepsilon$

What does "far" mean?
$\varphi(g)$ $(1-\varepsilon)$-far from $\varphi(e)$ $:\iff d(\varphi(g), \varphi(e)) \ge 1-\varepsilon$

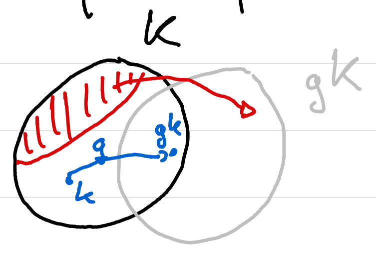
Examples

1) G residually finite, $F \subseteq G$ finite, $\varepsilon > 0$
$\exists \psi: G \to H$ finite, homom.: $\psi|_F$ inj.

$$\rightsquigarrow \begin{tikzcd} G \arrow[r, "\psi"] \arrow[rr, bend right, "\varphi"'] & H \arrow[r, "\mathrm{Cay}"] & \mathrm{Sym}(H) \end{tikzcd}$$

this works

2) G amenable, $e \in F \subseteq G$ finite, $\varepsilon > 0$.
By amenability $\exists K \subseteq G$ finite: $|FK| \le (1+\varepsilon)|K|$
Define $\varphi: G \to \mathrm{Sym}(|K|)$: $\varphi(g)(\underset{\in K}{k}) := gk$ whenever $gk \in K$
$\downarrow$ ext. doesn't matter?



What happens if $g \in F$?
$g \cdot K \subseteq F \cdot K \supseteq K \quad \Rightarrow |gK \cap K| \ge (1-\varepsilon)|K|$
Take $g, h, gh \in F$: $\exists K_{g,h,gh} \subseteq K$: $|K_{g,h,gh}| \ge (1-3\varepsilon)|K|$
$\forall k \in K_{g,h,gh}$: $\varphi(g)(k) = gk \qquad \varphi(h)(k) = hk \ldots$
$\exists$ large subset of K s.t. $\varphi(g)\varphi(h) = \varphi(gh)$
for all pts. in the subset

Other examples

G residually amenable $\Rightarrow$ G sofic

Definition due to Gromov/Weiss

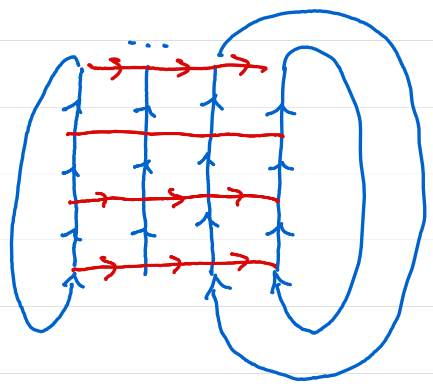
Conj. (Gromov?)
All groups are sofic.
Hyperbolic, Thompson, one-relator gps. ?

Let G be a fin. gen. gp., $G = \langle s_1, \ldots, s_n \rangle$
$\rightsquigarrow \pi: F_n \to G$, $N = \ker(\pi)$

Lemma:
G sofic $\iff \exists (\varphi_k: F_n \to \mathrm{Sym}(l_k))_{k \in \mathbb{N}}$
seqn. of homom. s.t.
1) $\forall g \in N$: $d(\varphi_k(g), 1_{l_k}) \to 0$
2) $\forall g \notin N$: $d(\varphi_k(g), 1_{l_k}) \to 1$

Example

$\mathbb{Z}^2 = \langle (1,0), (0,1) \rangle$



$\mathcal{U}$ ultrafilter

$$\prod_{k \in \mathbb{N}} \mathrm{Sym}(l_k) \Big/ \left((\sigma_k) \mid d(\sigma_k, 1_{l_k}) \xrightarrow[\mathcal{U}]{} 0 \right) = \mathrm{Sym}(\mathcal{U})$$

"ultraproduct"

$\rightsquigarrow G \xrightarrow{\Phi} \mathrm{Sym}(\mathcal{U})$ homom

Thm. (Elek–Szabo)

G sofic, K skew field,
then KG is finite $\quad (\Leftrightarrow a,b \in KG,\ ab=1 \Rightarrow ba=1)$
(pf. of Kaplansky's conj. for sofic gps.)

Pf.

G sofic $\Rightarrow \exists\, \varphi_k : \mathbb{F}_n \to \mathrm{Sym}(l_k)$ homom.
$G = \langle s_1, \dots, s_n \rangle$

1) $\forall r \in \ker(\pi : \mathbb{F}_n \to G)$: $d_{\mathrm{Sym}(l_k)}(\varphi_k(r), \mathbb{1}) \xrightarrow{k \to \infty} 0$

2) $\forall r \notin \ker(\pi : \mathbb{F}_n \to G)$: $\quad d_{\mathrm{Sym}(l_k)}(\varphi_k(g), \mathbb{1}_{l_k}) \xrightarrow{k \to \infty} 1$

$\mathrm{Sym}(l_k) \subseteq M_{l_k}(K)$
$\rightsquigarrow \varphi_m : K\mathbb{F}_n \to M_{l_m}(K)$

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$

$$M_{\mathcal{U}}(K) := \prod_{m \in \mathbb{N}} M_{l_m}(K) \Big/ \left((a_m) \,\middle|\, \frac{\mathrm{rk}(a_m)}{l_m} \xrightarrow{\mathcal{U}} 0\right)$$

1) $M_{\mathcal{U}}(K)$ behaves like a matrix algebra in many ways. In particular $\exists\, \mathrm{rk}_{\mathcal{U}} : M_{\mathcal{U}}(K) \to [0,1]$

i) $\mathrm{rk}_{\mathcal{U}}(a+b) \le \mathrm{rk}_{\mathcal{U}}(a) + \mathrm{rk}_{\mathcal{U}}(b)$

ii) $\mathrm{rk}_{\mathcal{U}}(ab) \le \min\{\mathrm{rk}_{\mathcal{U}}(a), \mathrm{rk}_{\mathcal{U}}(b)\}$

$$\mathrm{rk}_{\mathcal{U}}([(a_m)]) = \lim_{m \to \mathcal{U}} \frac{\mathrm{rk}(a_m)}{l_m}$$

2) $\mathrm{rk}_{\mathcal{U}}(a) = 1 \Leftrightarrow a$ inv. in $M_{\mathcal{U}}(K)$

$[(a_m)] \in M_{\mathcal{U}}(K)$, $\quad \frac{\mathrm{rk}(a_m)}{l_m} \xrightarrow[\mathcal{U}]{} 1$
Let b_m be a partial inverse,
$a_m b_m$, $b_m a_m$ idempotent of rank $\ge (1-\varepsilon_m) l_m$
$[(a_m b_m)] = 1 \quad \mathrm{rk}(1_{l_m} - a_m b_m) \le \varepsilon_m l_m$

Lemma: The normalised Hamming metric and the normalised rank are compatible in the following way:

$$\sigma \in \mathrm{Sym}(l) \qquad d(\sigma, \mathbb{1}_l) \underset{\substack{\text{univ.}\\ \text{constants}}}{\sim} \frac{\mathrm{rk}(\sigma - 1_{K^l})}{l}$$

$$d(\sigma, \mathbb{1}_l) = \frac{|\{ i \mid \sigma(i) \ne i \}|}{l}$$

von Neumann regular

$\rightsquigarrow$ We obtain

$$\begin{tikzcd} K\mathbb{F}_n \arrow[rr, "\Phi", "\prod \varphi_k"'] \arrow[dr, "\pi"'] & & M_{\mathcal{U}}(K) \\ & KG \arrow[ur, "\exists!"'] & \end{tikzcd}$$

since $\forall r \in \ker(\pi)$: $\Phi(r) = \Phi(e)$

Exercise: $KG \to M_{\mathcal{U}}(K)$ injective.

$\rightsquigarrow$ end of the proof.

"permutation"

P-stability

Let G be f.p. $G = \langle s_1, \dots, s_n \mid r_1, \dots, r_\ell \rangle$

$\varphi : \mathbb{F}_n \to \mathrm{Sym}(m)$ homom.

$$\mathrm{def}(\varphi) = \max_{1 \le i \le \ell} d(1_m, \varphi(r_i))$$

$$\mathrm{homdist}(\varphi) = \min_{\psi : G \to \mathrm{Sym}(m)} \max_{1 \le i \le n} d(\varphi(s_i), \psi(s_i))$$

Lemma:

$\exists c :\ \mathrm{def}(\varphi) \le c \cdot \mathrm{homdist}(\varphi)$

related to $\max\{\mathrm{length}(r_i) \mid \dots\}$

Def.:

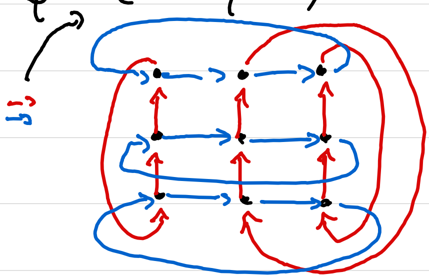
$G = \langle s_1, \dots, s_n \mid r_1, \dots, r_\ell \rangle$ is P-stable $\iff$

$\forall \varepsilon > 0 : \exists \delta > 0 : \mathrm{def}(\varphi) < \delta \implies \mathrm{homdist}(\varphi) < \varepsilon$

(indep. on pres.)

$G = \mathbb{Z}^2 = \langle a, b \mid [a,b] \rangle$

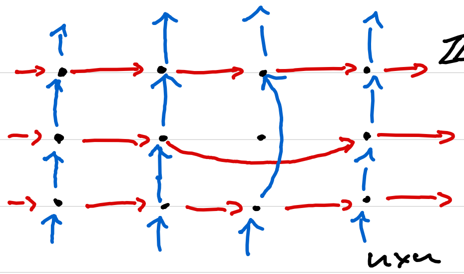
$\varphi : \mathbb{F}_2 \to \mathrm{Sym}(\ell)$ $\qquad$ $\mathrm{def}(\varphi)$ small



$\rightsquigarrow$ small defect

(weak stability: add a few pts. of a picture like)

(4)



$\mathbb{Z}^2 = \langle a, b \mid [a,b] \rangle$

$n \times n$ grid

$\#$ mistakes $\sim C$

$\#$ corrections $\sim n$

$\frac{1}{n^2} \#$ mistakes $\sim \frac{C}{n^2}$

$\frac{1}{n^2} \#$ corrections $\sim \frac{C}{n}$

Thm. (Arzhantzeva-Paunescu)
$\mathbb{Z}^2$ is P-stable.

Thm. (Becker-Lubotzky-Thom)
G amenable. TFAE
1) G P-stable
2) All invariant random subgps. of G are limits of f.i. inv. random subgps.

2) is true for polycyclic subgps.

Baumslag's group
$G = \langle a, t \mid a^2 = (tat^{-1})\, a\, (ta^{-1}t^{-1}) \rangle$

Thm. (Baumslag)
G is not res. fin.

Pf.
Claim: $G \overset{\pi}{\twoheadrightarrow} H$ finite, $\pi(a) = e_H$
$\langle \pi(a) \rangle \cong \mathbb{Z}/n\mathbb{Z}$
conj. by $\pi(tat^{-1})$ is mult. by 2.
$\rightsquigarrow 2^n \equiv 1 \mod n$
Exc.: this is not possible. :)

Consider $\pi(a)$ as a permutation on $\mathbb{Z}/n\mathbb{Z}$

t: $(1\ 2\ 3\ 4 \ldots) \to \underbrace{(1\ 2\ 4\ 8 \ldots)}_{\text{length} \geq \log_2(n)} (\ldots)(\ldots)(0)$

$\frac{n}{\log_2(n)}$ "no. of mistakes"

$\rightsquigarrow G$ sofic.

Central extensions

Λ res. fin. gp.

$1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma \overset{s}{\rightleftarrows} \Lambda \to 1$ central extension

$\leadsto [\alpha] \in H^2(\Lambda; \mathbb{Z}/2\mathbb{Z})$

Fact 1: $\Lambda' < \Lambda$ finite index

$\leadsto \quad 1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma' \to \Lambda' \to 1$

Γ not res. fin. $\Rightarrow$ Γ' not res. fin.

$\Downarrow \Gamma' \not\cong \Lambda' \times \mathbb{Z}/2\mathbb{Z}$

$$\begin{array}{ccc}
[\alpha] \in H^2(\Lambda; \mathbb{Z}/2\mathbb{Z}) & & \\
\downarrow & & \downarrow \\
\text{not } 0 & & H^2(\Lambda'; \mathbb{Z}/2\mathbb{Z}) \\
& & \Vert\!\wr \;\leftarrow \text{Shapiro} \\
[\alpha_{\hat{\Lambda}}] \in H^2(\Lambda, M(\hat{\Lambda}, \mathbb{Z}/2\mathbb{Z})) & \longleftarrow & H^2(\Lambda; \mathrm{Fun}(\Lambda/\Lambda', \mathbb{Z}/2\mathbb{Z}))
\end{array}$$

$\uparrow$ meas. fns.

induced by $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Fun}(\Lambda/\Lambda', \mathbb{Z}/2\mathbb{Z})$ as constant functions

Thm.:

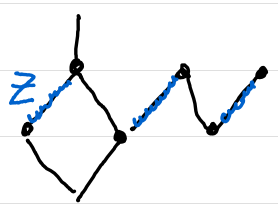
$\Lambda \circlearrowright E\Lambda$, $B\Lambda = E\Lambda/\Lambda$ high dim. $\leadsto [\alpha_{\hat{\Lambda}}] \neq 0$

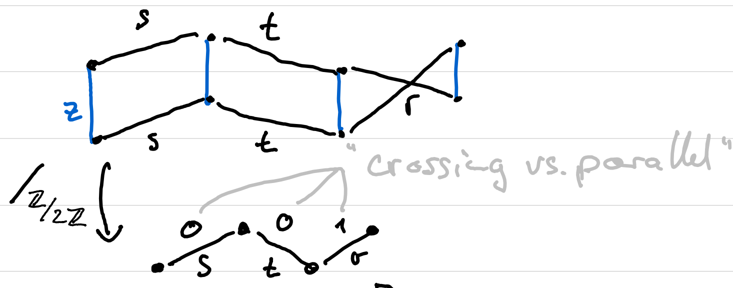
Let's now assume Γ is sofic.

$\langle z \rangle = \mathbb{Z}/2\mathbb{Z}$

$\Lambda = \langle S \mid R \rangle$

$\Gamma = \langle \tilde{S}, z \mid \tilde{R}, z^2, [s,z] \rangle$





sofic approx. of Λ

Limit actions

Suppose $\varphi_k : \mathbb{F}_s \to \mathrm{Sym}(l_k)$ is a sofic approx. of Λ

Let X be the cpct. T_2-space s.t.

$$C(X) = \ell^\infty\left(\coprod_{k\geq 1}\{1,\dots,l_k\}\right) \Big/ \ker(\alpha)$$

$$\alpha : \ell^\infty\left(\coprod_{k\geq 1}\{1,\dots,l_k\}\right) \longrightarrow \mathbb{C}$$

$$f \longmapsto \lim_{k\to\mathcal{U}} \frac{1}{l_k}\sum_{i=1}^{l_k} f(i\in\{1,\dots,l_k\})$$

$$\ker(\alpha) = \{f \mid \alpha(|f|) = 0\}$$

"all mistakes lie in this kernel"

Lemma:

$\exists$ canonical action $\Lambda \curvearrowright (X,\mu_\alpha)$ p.m.p.

$$H^2(\Lambda, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^2(\Lambda, M(X,\mathbb{Z}/2\mathbb{Z}))$$

$$[\alpha] \xrightarrow{\text{Thm.}} [\alpha_X] = 0$$

Cor.

If Γ is sofic $\Rightarrow$ Λ is not P-stable