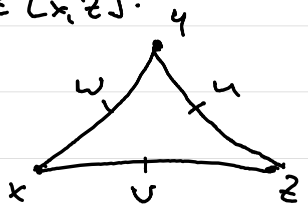
Minicourse: Boundaries of hyperbolic and relatively hyperbolic groups

From 0 to infinity

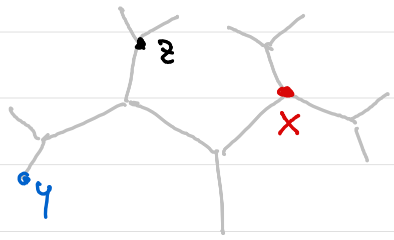
X proper geod. metric space

Lemma/Def. (Gromov product)
xyz a (geodesic) triangle in X
$\exists\, w \in [x,y],\ u \in [y,z],\ v \in [x,z]$:
$d(x,w) = d(x,v)$
$d(y,w) = d(y,u)$
$d(z,v) = d(z,u)$



$$(y|z)_x = (y,z)_x = \frac{1}{2}\left(d(x,y) + d(x,z) - d(y,z)\right)$$

Ex.: In a tree:



$$(y|z)_x = 2$$

Def.: For some $\delta \geq 0$
X is δ-hyp. if for any $\Delta\, xyz$ and
$y' \in [x,y],\ z' \in [x,z]$ with $d(x,y') = d(x,z') \leq (y,z)_x$:
$$d(y',z') < \delta$$

Def.:
X is <u>Gromov-hyperbolic</u> if $\exists \delta$: X is δ-hyperbolic

Ex.:
X is Gromov-hyperbolic iff
$\exists \delta: \forall \Delta\, xyz: [x,y] \subseteq N_\delta([x,z]) \cup N_\delta([y,z])$

Ex.:
$\mathbb{H}^2$ is Gromov-hyperbolic

What about groups?

A gp. is (Word Gromov) hyperbolic if it acts geometrically on a (proper, geodesic) metric space

↳ prop. disc., cocptly, by isometries

$f: X \to Y$ is a quasi-isometric map (q-i emb.)
if $\exists\, a \geq 1, b \geq 0$ s.t.
$\forall x, x' \in X: \ \frac{1}{a} d(x,x') - b \leq d(f(x), f(x')) \leq a\, d(x,x') + b$

cobounded

$f: X \to Y$ is quasi-dense ($f(X)$ is a net)
if $\exists\, c \geq 0: \ Y \subseteq N_c(f(X))$

f is a quasi-isometry if it is a q-i emb. and $f(X)$ a net.

<u>Ex.:</u>

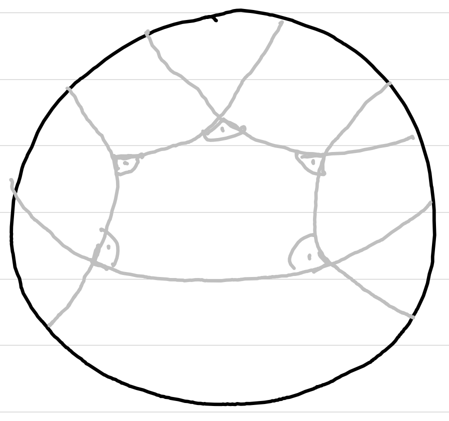
Then $\exists\, g: Y \to X$: $f \circ g$ and $g \circ f$ are bdd dist. from Id.

Stability of quasi-geodesics (Morse lemma)

If X is a δ-hyperbolic space, $c:[a,b]\to X$ a $(2,\varepsilon)$ quasi-geodesic from p to q and $[p,q]$ a geodesic in X,
$$d_{Haus}([p,q], \mathrm{im}(c)) \le R = R(\delta, 2, \varepsilon)$$
Hausdorff dist. (geodesics are close)

Ex.:
You can use this to show that X is Gromov-hyp. if it is q-i to a Gromov-hyp. space.
If G acts geometrically on some hyperbolic space, then any space it acts on geom. is also hyperbolic.

G is hyperbolic if
G is fin. gen. by S and $\mathrm{Cay}(G,S)$ is hyp.

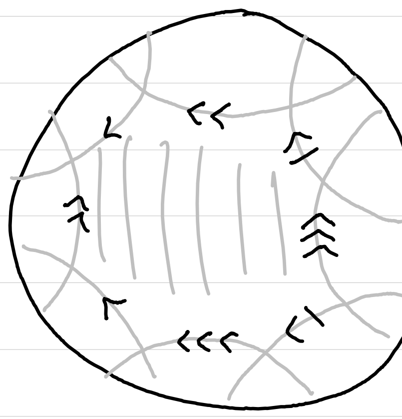


right angled pentagon in $\mathbb{H}^2$

images of refl. along sides $\to$ tile $\mathbb{H}^2$

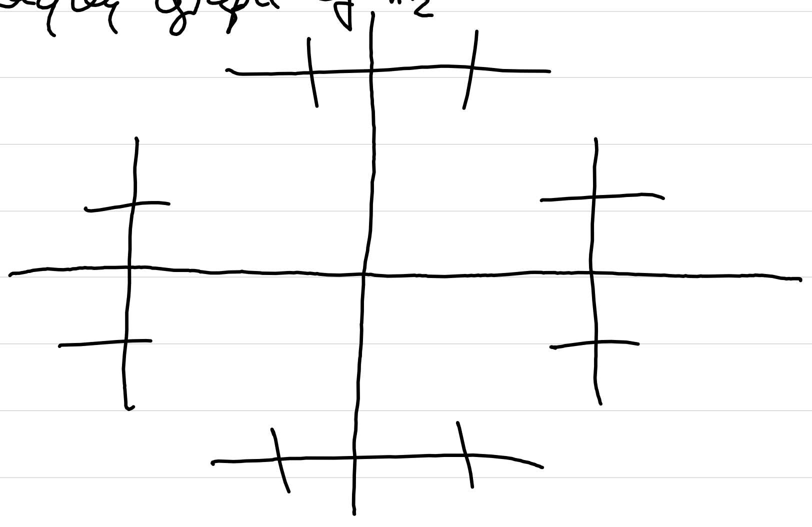
$\to$ this group acts geom. on $\mathbb{H}^2$
This is a hyperbolic gp.

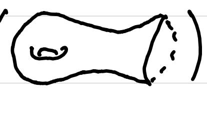
right angled Coxeter gp. (RACG) [(V,E) graph?]
$\langle v_i \in V,\ v_i^2 = 1 \mid v_i v_j = v_j v_i \text{ if } (i,j) \in E$

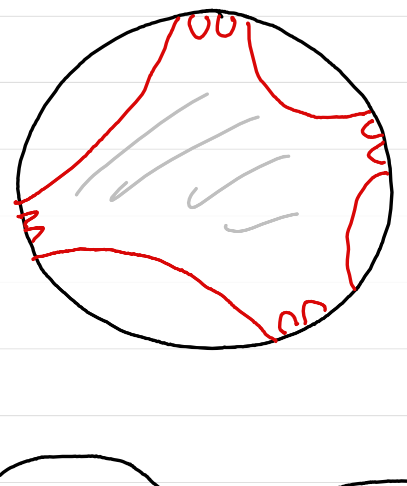


$\exists$ a hyp. octagon w/ angles $\frac{2\pi}{8}$

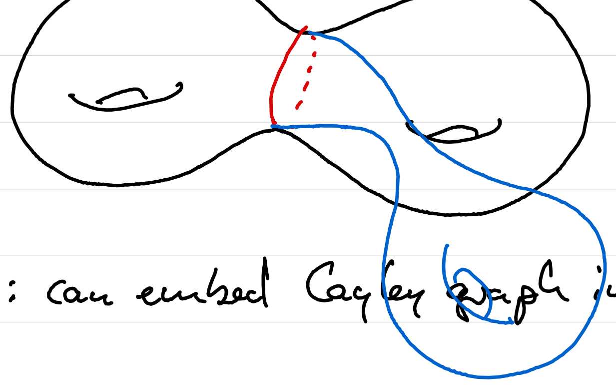
Cayley graph of $\mathbb{F}_2$ (free gp. on 2 gen.)



$\to \pi_1($  $)$



convex subset of $\mathbb{H}^2$ $\to$ δ-hyp.

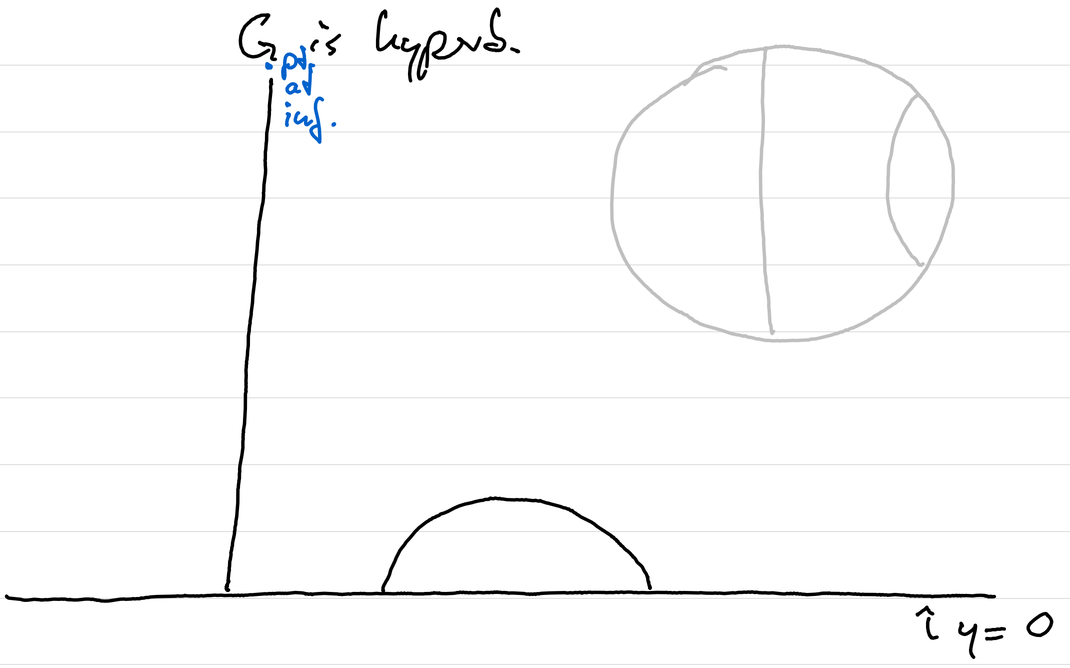


(Exc.: can embed Cayley graph in $\mathbb{H}^2$)

OK, what about the boundary?

X hyperbolic, ∂X a top. obj.
It's canon. assoc. to X.

$\to$ compactification
$\to$ if X q-i to Y, then their bdaries are homeomorphic.
this allows us to talk about ∂G if G is hyperb.



$\exists$ Defs.

X hyperb. proper geod. metric sp.

I $\partial X = \{\text{geod. rays from } x_0\} / \sim$

$\sim$: they stay close together:
$\exists c: \operatorname{im}(\gamma) \subseteq N_c(\operatorname{im}(\gamma'))$
$\operatorname{im}(\gamma') \subseteq N_c(\operatorname{im}(\gamma))$

II $\partial X := \{\text{q-geod. rays}\} / \sim$

$\sim$ as before

Fact: in a Gromov-hyp. space X,
$\exists \delta : (x|y)_p \leq d(p,[x,y]) \leq (x|y)_p + \delta$
"the Gromov-prod. appr. the dist of p to $[x,y]$"

III sequences "tending to ∞" up to equivalence
(x_i) sequence in X
$(x_i) \to \infty$ if $\lim_{i,j\to\infty} (x_i|x_j)_p \to \infty$

(Exc.: indep. of p)

Equivalence: $(a_i) \sim (b_i)$ if $\liminf_{i\to\infty} (a_i, b_i)_p = \infty$

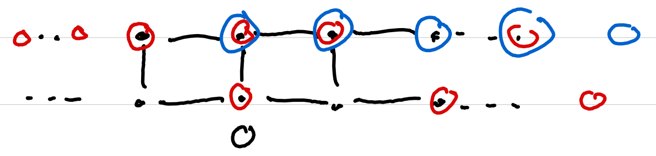
Rough idea why they are the same:
geod. ray at $x_0 \in \{\text{quasi-geod. rays}\}$
γ q-geod. ray $\rightsquigarrow (\gamma_i)_{i\in\mathbb{N}}$ sequ. tending to ∞

Topology:
$x_i \to m$ for m an equ. class as in III
$m = [x_i]$

$$(m|n)_w = \sup_{\substack{x_i \to m \\ y_i \to n}} \liminf_{i\to\infty} (x_i|y_i)_w$$

Exc.:

$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$



Exc.: play w. examples

(Day 2)

Topology on the boundary

X hyp. proper geod. metric space

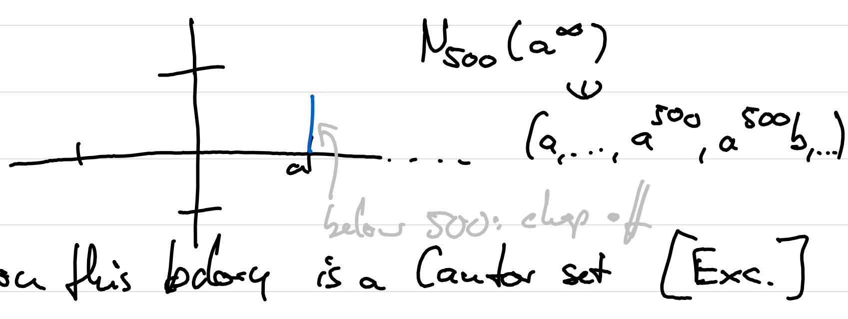
$x \in \partial X$

$N_r(x) = \{ y \mid (x,y)_w > r \}$

$y \in \partial X$: $(x,y)_w = \sup_{\substack{x_i \to x \\ y_i \to y}} \{ \liminf (x_i, y_i)_w \}$

$y \in X$: $(x,y)_w = \sup_{x_i \to x} \{ \liminf (x_i, y)_w \}$

F_2



Topology on this bdary is a Cantor set [Exc.]

The bdary is a q-i invariant

$f: X \to X'$ is a q-i between hyp. metric sp.

$\partial X = $ q-geod. / bded Hausd. dist.

$\partial f: \partial X \to \partial X'$ def. by $[\gamma] \to [f(\gamma)] \in \partial X'$

$\to$ doesn't dep. on choice of γ:

$\gamma_1 \sim \gamma_2 \Rightarrow d_H(\gamma_1, \gamma_2) < c$

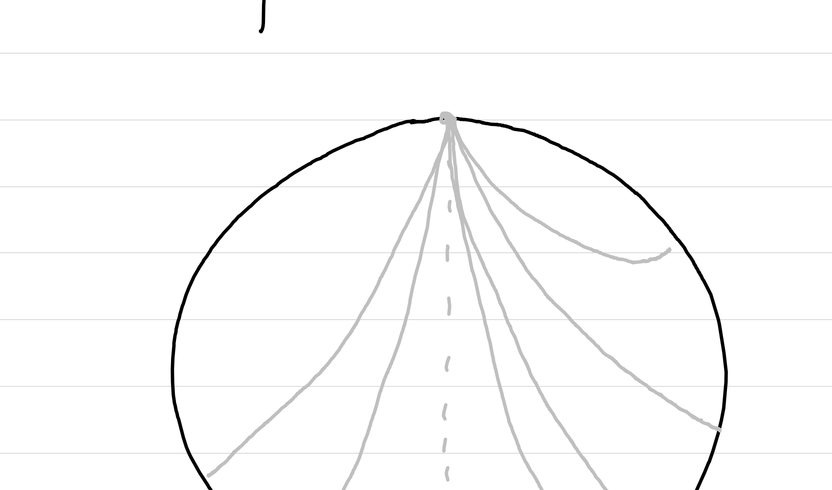
$\Rightarrow d_H(f(\gamma_1), f(\gamma_2)) \leq ac + b$

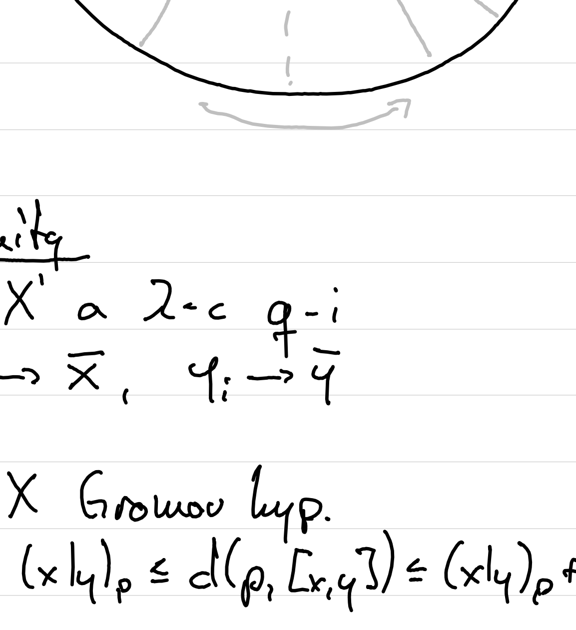
$d_H((g \circ f)(\gamma), \gamma)$ is bounded, so $[(g \circ f)(\gamma)] = [\gamma]$

$\rightsquigarrow \partial g \circ \partial f = \mathrm{Id}_{\partial X}$, $\partial f \circ \partial g = \mathrm{Id}_{\partial X'}$

Note: A gp. acting by isometries on X also acts on ∂X

$z \mapsto z+1$





Continuity

$f: X \to X'$ a λ-c q-i

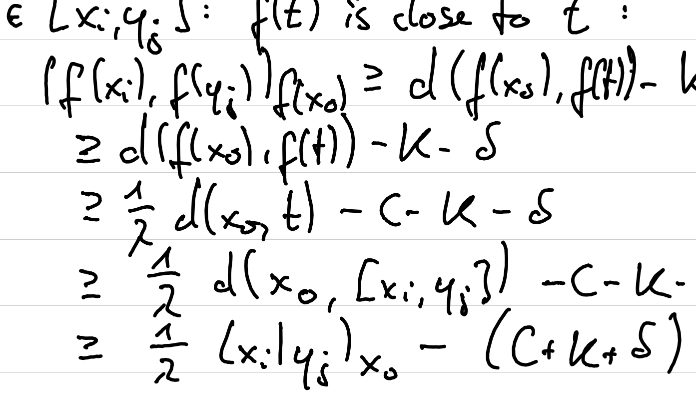
$x_i \to \bar{x}$, $y_i \to \bar{y}$

Fact: X Gromov hyp.

$\exists \delta: (x|y)_p \leq d(p, [x,y]) \leq (x|y)_p + \delta$

In X': $(f(x_i), f(y_j))_{f(x_0)} \geq d(f(x_0), [f(x_i), f(y_j)]) - \delta$

$= d(f(x_0), t') - \delta$



for some $t' \in [f(x_i), f(y_j)]$

$[x_i, y_j]$ maps to a q-geodesic

$\exists t \in [x_i, y_j]$: $f(t)$ is close to t':

$(f(x_i), f(y_j))_{f(x_0)} \geq d(f(x_0), f(t)) - K - \delta$ for some K

$\geq d(f(x_0), f(t)) - K - \delta$

λ-C q-i $\quad \geq \frac{1}{\lambda} d(x_0, t) - C - K - \delta$

$\geq \frac{1}{\lambda} d(x_0, [x_i, y_j]) - C - K - \delta$

$\geq \frac{1}{\lambda} (x_i | y_j)_{x_0} - (C + K + \delta)$

So, if $(x_i | y_j)_{x_0} > \lambda(n + C + K + \delta)$

then $(f(x_i) | f(y_j))_{f(x_0)} > \frac{1}{\lambda}(\lambda(n + C + K + \delta)) - (C + K + \delta) = n$

q-i $\Rightarrow$ bij. map

also cts. map between cpct. T_2-spaces $\Rightarrow$ homeom.

Why is this great? $\quad G$ hyperbolic

G acts geom. on X, Y

$\to \partial X \cong \partial Y \to$ We can define $\partial G = \partial X \cong \partial Y$

This opens the field:

Look at ∂X, what possible G have this bdary?

$\partial G = 2$ pts. $\to$ G virtually $\mathbb{Z}$

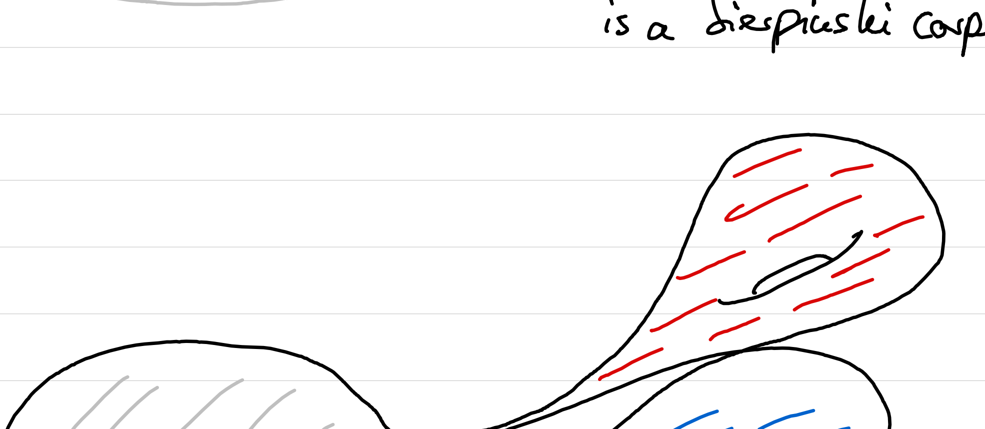
$\partial G = $ Cantor set $\to$ G virt. free (rank ≥ 2)

$\partial(\pi_1(S_g)) = S^1$:

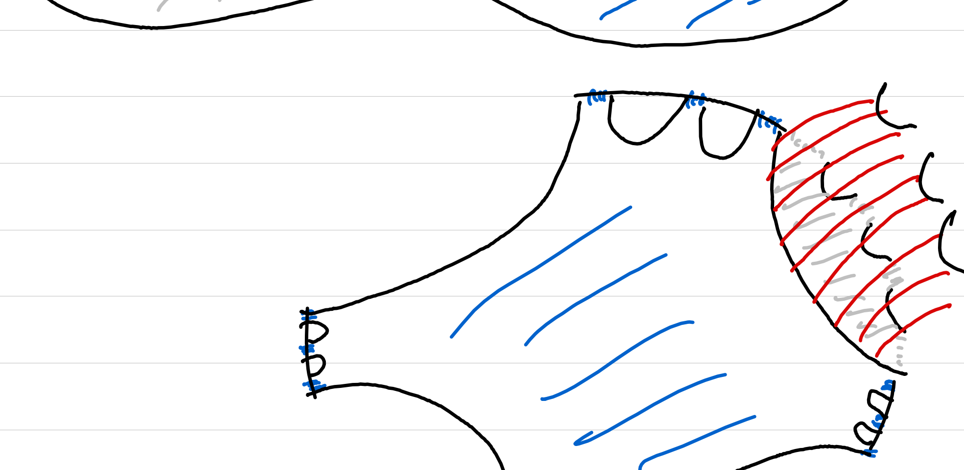
Thm (Tukia, Gabai, ...)

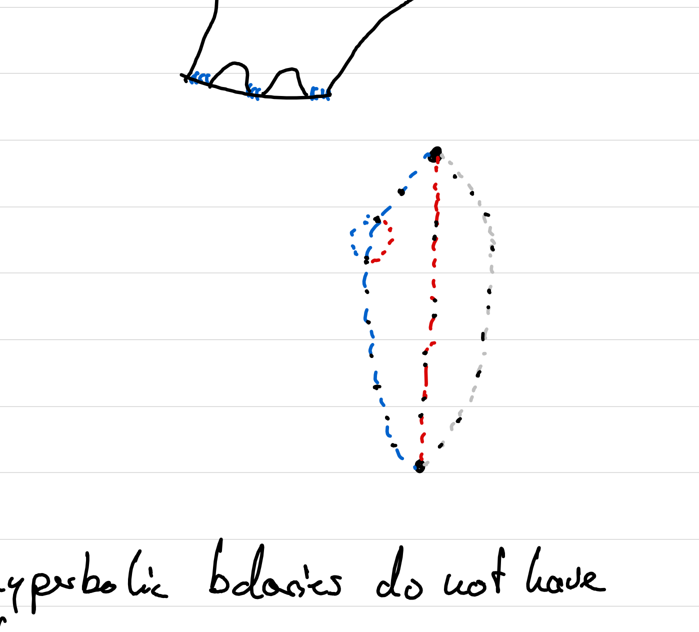
If $\partial \Gamma \cong S^1$, then Γ is virt. Fuchsian

What other boundaries can occur?



$\partial(\pi_1(M^3$ w/ tot. geod. bd.$)$ is a Sierpinski carpet





Fact: com. hyperbolic bdaries do not have cut pts.

Bowditch: local cut pts. $\longleftrightarrow$ splittings over $\mathbb{Z}$

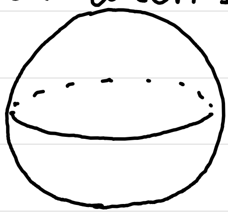
$$\begin{tikzcd}
 & \mathbb{Z} * \mathbb{Z} \arrow[d, no head] & \\
 & \mathbb{Z} \arrow[dl, no head] \arrow[dr, no head] & \\
\mathbb{Z} * \mathbb{Z} & & \mathbb{Z} * \mathbb{Z}
\end{tikzcd}$$

Thm. (Bowditch)

This splitting yields vertex stabilisers

- elementary ($\mathbb{Z}$) (<3 pts. in bdary)
- hanging Fuchsian
- rigid (don't admit more splittings over $\mathbb{Z}$)

So far, the examples of ∂G have been Kleinian.
Kleinian: discr. subgp. of $PSL(2,\mathbb{C}) \cong \mathrm{Isom}^+(\mathbb{H}^3)$



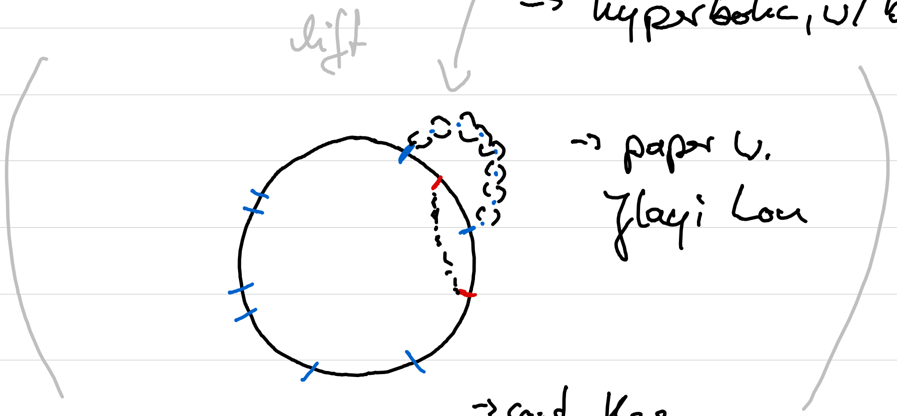
For nice subgps. $H < \mathrm{Isom}^+(\mathbb{H}^3)$, ∂H is $\underbrace{\Lambda H}_{\overline{\text{orbit}(x) \text{ under } H} \cap \partial \mathbb{H}^3}$ limit pts.

For Kleinian gps., the bdary is planar.
(because ∂H embeds in S^2)

Let's look at a gp. whose boundary is not planar.

can be realised as a ~~3-mfd.~~

π_1 of $\to$ is π_1 of a 3-mfd.
$\to$ hyperbolic, w/ bdary



$\to$ paper w. Zlayi Lou

$\to$ cont. $K_{3,3}$
$\to$ can not be emb. in S^2

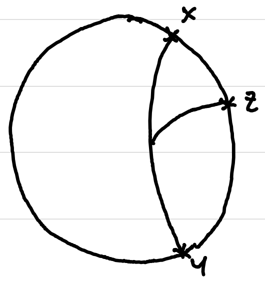
Discrete gps. in $PSL(2,\mathbb{C})$ are cool in that they are convergence gps. (Gehring, Martin)

__Def.__ (discrete) convergence gp. on M
Let M be a cpct. T_2-space. A gp. of homeoms. of M is a __convergence gp.__ if
(g_i) distinct $\Rightarrow \exists (g_{n_i}); a, b$ on M: $g_{n_i} \to a$ unif. on cpct. sets in $M \setminus \{b\}$

Tukia, Freden

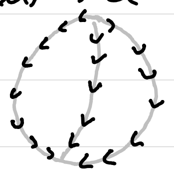
A hyperbolic gp. acts as a convergence group on its boundary

A convergence gp. action on $M \Rightarrow$
properly disc. action on $T = (x,y,z) \in M^3$, not all equal
$x \neq y \neq z$?



Tukia: A conv. gp. has a $T/_G$ cpct. iff every pt. of M is a conical limit pt for the action of G

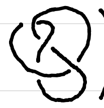
conical limit pt.:
$m \in M$ is conical if $\exists (g_i) \subset G$, $a \neq b$:
$g_i(m) \to a$ but $g_i(x) \to b$ for $x \neq m$



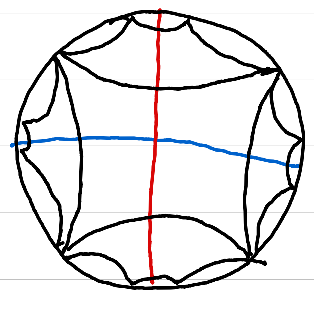
Bowditch

$G \curvearrowright M$ as a conv. gp. is hyperbolic iff every pt of M is a conical limit pt.
G is hyperbolic + $\partial G = M$

This not all that happens dynamically on $\mathrm{Isom}(\mathbb{H}^3)$
$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \right\rangle$ ($\mathbb{C} \times \mathbb{R}^+$ upper half space)
↳ fixes ∞
∞ is not a conical limit pt.

$\pi_1 (S^3 \setminus$  $)$ hyperbolic knot complement
There is a gp. of $\mathrm{Isom}^+(\mathbb{H}^3)$, Γ,
where $\mathbb{H}^3/_\Gamma$ has fin. volume.
not a hyperbolic gp.: $\mathbb{Z} \oplus \mathbb{Z}$ & hyp gp

Ex.



(nice pictures in paper)
$[a,b]$ is parabolic
→ fixes one pt. on bdry
This is a rel. hyp. gp. pair
$(\mathbb{F}_2, [a,b])$

$\pi_1 \Bigg($  $\Bigg) = \mathbb{F}_2$

Relatively hyperbolic gp. pair (Bowditch, ~ Farb)
$(G, \mathcal{P})$ is relatively hyperbolic if
G acts prop. disc. by isom. on the proper hyperbolic space X
s.t. every pt. of ∂X is either
a) a conical limit pt.
b) a bounded parabolic pt.

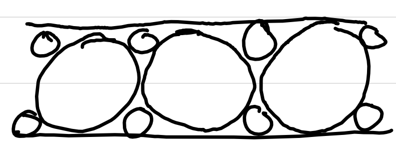
m is parabolic if
$\exists$ parab. subgp. fixing m $P_m < \mathcal{P}$ +
$\left(\partial X \setminus m \big/ P_m \text{ is cpct.} \right) \to$ "bdd."
+ $\mathcal{P}$ is the collection of maximal parab. subgps.

$\partial(G, \mathcal{P}) = \partial X$

- It is <u>not</u> in general true that: if $(G, \mathcal{P})$ acts rel. hyp. on X, Y, then X, Y are q-i.

Bowditch: But, $\partial X \cong \partial Y$
So this bdary is well-def. for $(G, \mathcal{P})$

$\partial(\mathbb{F}_2, \langle [a,b] \rangle) = S^1$
$\partial(\mathbb{F}_2, \{\langle a,b \rangle, \langle a \rangle, \langle b \rangle\}) \to$



today: • conn. between hyperbolic + rel. hyperb. bdaries
• planar bdaries of rel. hyp. gps. / conj.
• more examples of cool bdaries

M compact metrisable perfect compactum (S^2)
no isol. pts.

G acts on M as a conv. gp. where all pts. are conical $\iff$ G hyp. + $\partial G \cong M$ / Bowditch Tukia

This goes through the space T of dist. triples of M.
(x,y,z) pairw. dist? vs. $x \neq y \neq z$

Action is prop. disc. on T when G acts as a conv. gp.
T/G is cpct. $\iff$ pts. of M are conical

Analog of this for rel. hyp. gps.:
(Asli Yaman) M compact, metrisable, perfect:
Γ acts on M as a GF conv. gp. (all pts. are conical or bded. parab. ($p \in \mathcal{P}$ are f.g.))

$\Updownarrow$

$(\Gamma, \mathcal{P})$ is rel. hyp. $\partial(\Gamma, \mathcal{P}) = M$

And $(\Gamma, \mathcal{P})$ acts nicely on T.

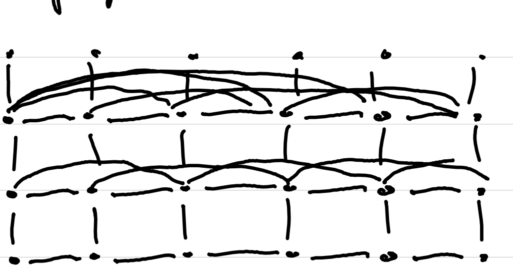
Another way to get a space

Bowditch, Cooper

[drawing]

Version: Groves, Manning
Sisto

Γ graph $\mathcal{H}(\Gamma)$



$V(\mathcal{H}(\Gamma)) = V(\Gamma) \times \mathbb{Z}_{\geq 0}$

$E(\mathcal{H}(\Gamma)) = \{[(v,k),(w,k)] ; \{ [v,w] \in E(\Gamma) \} \cup$
$\{[v,k], \text{ if } d_\Gamma(v,w) \leq 2^k\} \cup \{[(v,k),(v,k+1)]\}$

Cusped profile graph $X(G,\mathcal{P})$
G a gp., $\mathcal{P}$ a fin. coll. of subgps.
$\mathcal{A}$: gen. set that contains a gen. set for $P \in \mathcal{P}$

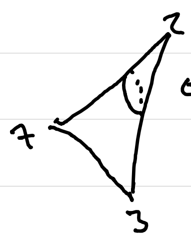
- $C(G,\mathcal{A})$
- attach $\mathcal{H}(gP)$ to each coset gP

Groves-Manning: $(G,\mathcal{P})$ is rel. hyp. when $X(G,\mathcal{P})$ is hyp.
$\partial X(G,\mathcal{P}) = \partial(G,\mathcal{P})$

Analog of this for rel. hyp. gps.
Splitting is <u>rel. to $\mathcal{P}$</u> if each subgp. in $\mathcal{P}$ is conj. into one of the vertex gp.

[drawing]

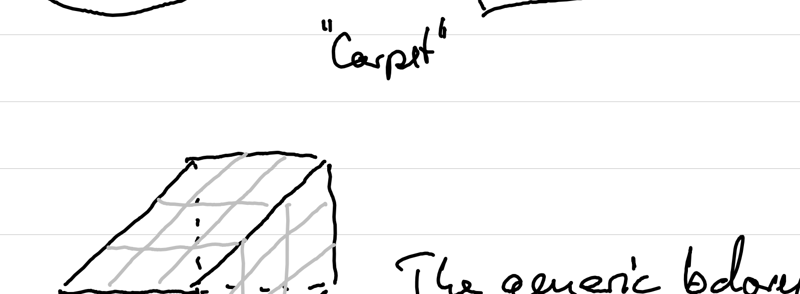
Bowditch:
local cut pts. in ∂G
$\Updownarrow$
splittings over 2-ended gps.
except: when $\partial G = S^1$ and gp. is rigid



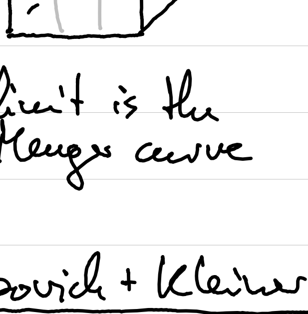
← does not have any splittings

Haulmark proved an analog of Bowditch's thm.
(let's assume no cut pts.)
local cut pts. in $\partial(G,\mathcal{P})$ $\longleftrightarrow$ splittings rel $\mathcal{P}$ over virt. cyclic gps.
$\partial(G,\mathcal{P}) \neq S^1$

We need to understand the rigid pieces
Bdaries w/ no local cut pts. (boundaries of vertex subgps.)



"Carpet"



The generic bdary:
Most hyperbolic gps. have this as ∂G!!

limit is the Menger curve

<u>Kapovich + Kleiner:</u>
1-ended hyperbolic gp. with ∂G 1-dim. + G does not split over a 2-ended gp., then ∂G:
1) S^1
2) Carpet
3) Menger curve

Kapovich, Kleiner conj. that if $\partial G \cong$ Carpet, then G is virtually $\pi_1(M^3)$, M^3 hyp. mfd. w/ t.g. bdary

Haulmark (Dasgupta): $(G,\mathcal{P})$ rel. hyperbolic, $\dim(\partial(G,\mathcal{P}))=1$, $P \in \mathcal{P}$ one-ended + G does not split rel $\mathcal{P}$ over 2-ended gps
Then, $\partial(G,\mathcal{P}) =$
- circle
- Sierpiński carpet
- Menger curve

Conj.

Cannon: G hyp., acts effectively on its bdary, $\partial G \cong S^2 \Rightarrow G$ is a subgp. of $\mathrm{Isom}^+(\mathbb{H}^3)$

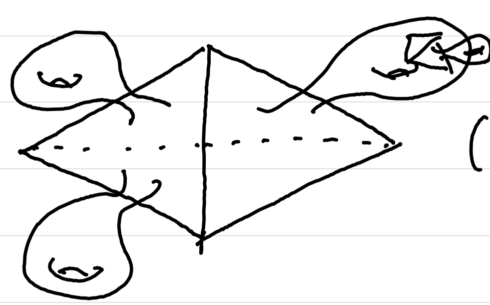
(G is virt. $\pi_1(M^3)$)

Cannon (Relative): (Groves-Manning-Sisto) (Tshishiku-Walsh)

$(G, \mathcal{P})$ rel. hyp., G torsion free

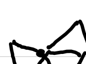
If $\partial(G, \mathcal{P}) \cong S^2$, then G is $\pi_1(M^3)$

$M^3 =$ hyperbolic 3-mfd., either finite vol. or hyperbolic w/ f.g. bdary



(triang. S^2, put genus, triang., ...)

This is the bdary of a RACG.

 graph has flagification