① Lecture 4 Genevieve Walsh

- connections between hyperbolic and relatively hyp boundaries
- Planar boundaries of rel hyp groups / conjectures
- More examples of cool boundaries

$M$ cpt, metrisable perfect (no isolated pts) compactum ($S^?$)
$G$ acts on $M$ as a convergence gp where all points are conical. $\Longleftrightarrow$ $G$ hyperbolic $\partial G \cong M$

This goes through the space $T$ of distinct triples of $M$ $T = (x, y, z)$ $x \neq y \neq z$

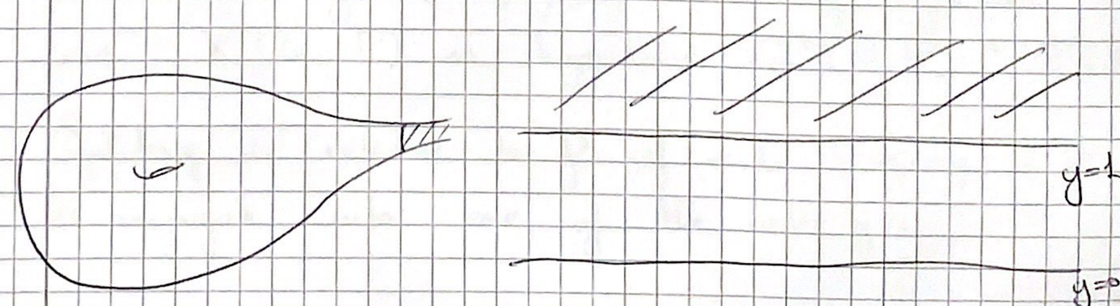
Action is properly discontinuous on $T$ where $G$ acts as a convergence gp $G \backslash T$ cpt $\Longleftrightarrow$ pts of $M$ are conical

Analog of this for relatively hyp groups
(Asli Yaman) $M$ cpt, metrisable, perfect

$\Gamma$ acts on $M$ as a GF convergence gp (all pts are conical or bdd parabolic ($P \in \mathcal{P}$ are f.g))
$\Longleftrightarrow$ $(\Gamma, \mathcal{P})$ is relatively hyperbolic $\partial(\Gamma, \mathcal{P}) = M$.

And $(G, \mathcal{P})$ acts nicely on $T$.

Another way to get a space

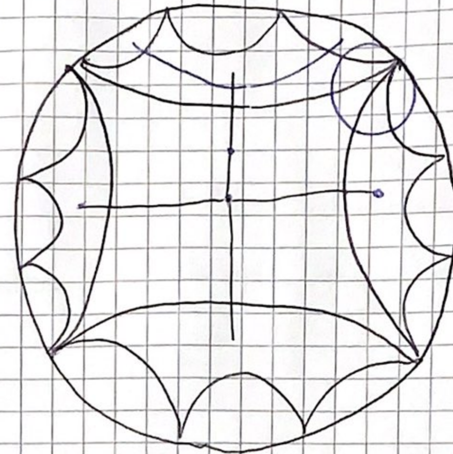


(2)

Bowditch, Cooper

version: Groves + Manning

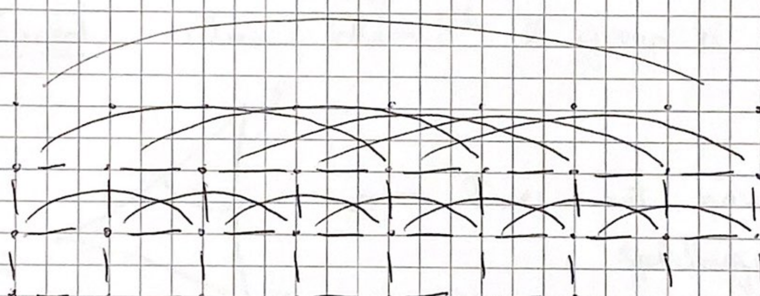
Sisto



$\Gamma$ graph $\mathcal{H}(\Gamma)$ $\quad V(\mathcal{H}(\Gamma)) = V(\Gamma) \times \mathbb{Z}_{\geq 0}$

$E(\mathcal{H}(\Gamma)) = [(v,k),(w,k)]$ if $[v,w] \in E(\Gamma)$ and $[(v,k),(w,k)]$ if $d_\Gamma(v,w) \leq 2^k$

vertical edges $[(v,k),(v,k+1)]$



Cusped Profile graph $X(G, \mathbb{P})$. $G$ group, $\mathbb{P}$ finite collection of subgroups. $A$ generating set which contains a generating set for $P \in \mathbb{P}$.

- $C(G, A)$
- attach $\mathcal{H}(gP)$ to each coset $gP$.

Groves - Manning $(G, \mathbb{P})$ is relatively hyperbolic when $X(G, \mathbb{P})$ is hyperbolic $\partial X(G,\mathbb{P}) = \partial(G,\mathbb{P})$

Splitting is relative to $\mathbb{P}$ if each subgroup in $\mathbb{P}$ is conjugate into one of the vertex gps.

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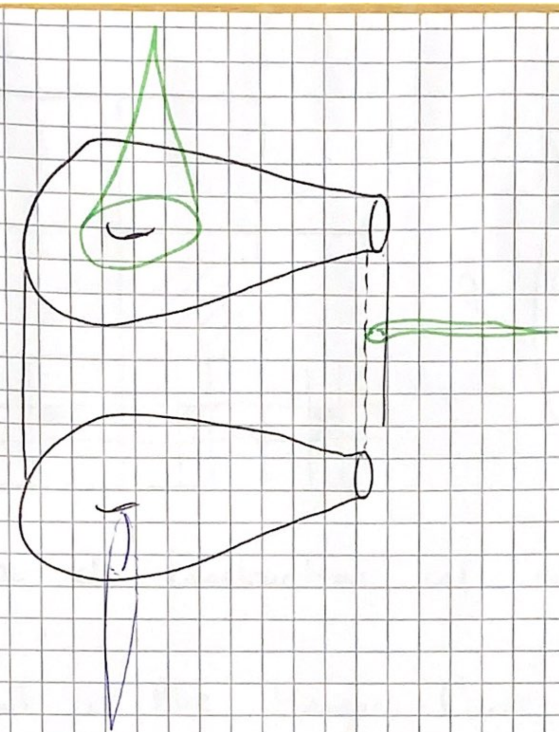
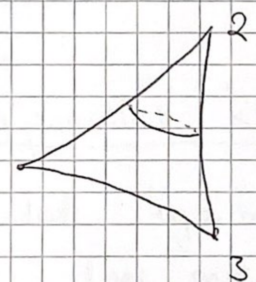


This does not have any splitting rel $\mathcal{P}$.

Bowditch: local cut points in $\partial G \iff$ splittings over 2-ended gps.

Except: when $\partial G = S^1$ & group is rigid.




Does not have any splittings

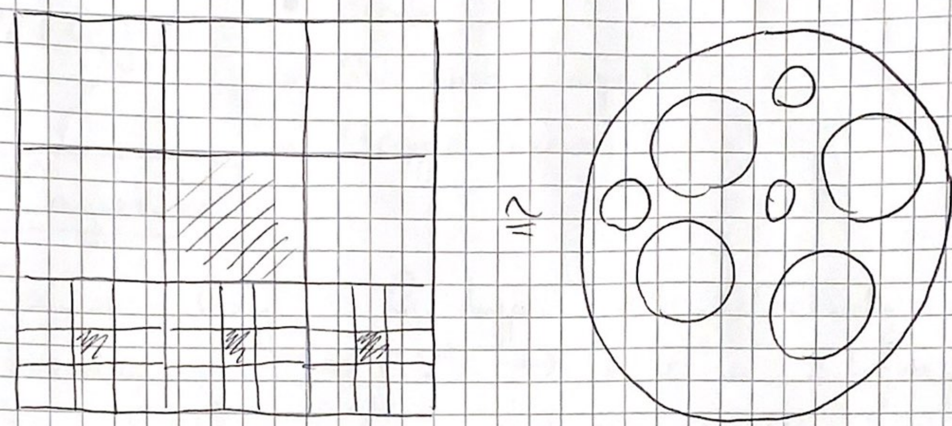
Hruska-Haulmark proved an analogue of Bowditch's thm. (let's assume no cut points)

local cut points in $\partial(G, \mathcal{P}) \iff$ splittings rel $\mathcal{P}$ over virtually cyclic gps

$\partial(G, \mathcal{P}) \neq S^1$

We need to understand the rigid pieces.

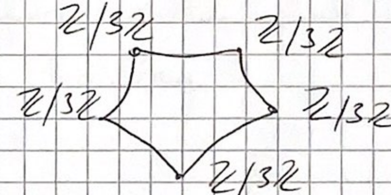
Boundaries w/ no local cut points



Can do construction w/ cube 

Generic!

limit is the Menger Curve. Most hyperbolic groups have this as $\partial G$! (eg. boundary of a hyperbolic building)




## Kapovich + Kleiner

1-ended hyperbolic group with $\partial G$ 1-dimensional + $G$ does not split over a 2-ended group with

$\partial G$ 1. $S^1$ 2. Carpet 3. Menger curve.

Kapovich + Kleiner conjectured: if $\partial G \simeq$ 

Then $G$ is virtually $\pi_1(M^3)$, $M^3$ hyp manifold w/ totally geodesic boundary

+ dim $\partial = 1$

Haulmark (Dasgupta) $(G, \mathcal{P})$ relatively hyp $P \in \mathcal{P}$ one ended, $G$ does not split rel $\mathcal{P}$ over 2-ended gps. Then $\partial(G, \mathcal{P})$

$\partial(G, \mathcal{P}) =$ 
- circle
- Sierpinski carpet
- Menger curve

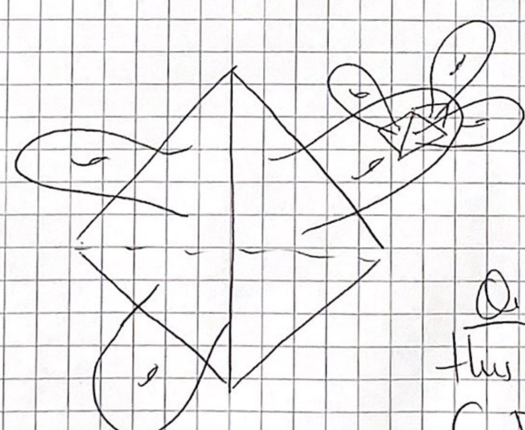
$\boxed{\text{Conjectures}}$

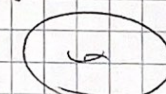
<u>Cannon Conj</u>: $G$ hyp, acts effectively on its boundary, $\partial G \cong S^2 \Rightarrow G$ is a subgroup of $\mathrm{Isom}^+ \mathbb{H}^3$ ($G$ is virtually $\pi_1(M^3)$)

<u>Relative Cannon Conjecture</u> (Groves - Manning - Sisto) (Tshishiku - Walsh)
$(G, \mathcal{P})$ relatively hyperbolic $G$ torsion-free
If $\partial(G, \mathcal{P}) \cong S^2$, then $G$ is $\pi_1(M^3)$
$M^3$ = hyperbolic 3-manifold either finite volume or hyperbolic w/ tg boundary

<u>One last example!</u>

This is the boundary of a RACG



 graph has flagification 

<u>Q:</u> which groups have this as a boundary? (Pontryagin surface)