Conjectures: $\boxed{\text{Cannon conj.}}$: G hyp, acts effectively on its boundary + $\partial G \simeq S^2$

$\Rightarrow$ G is a subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$
(G is virt. $\pi_1 M^3$)

Relative Cannon conjecture: (Groves, Manning, Sisto; Tshi...-W)

(G, P) relatively hyperbolic G torsion-free

If $\partial(G,P) \cong S^2$, then $G = \pi_1(M^3)$

with M^3 = hyp. 3-mnfld either finite volume or hyperbolic w/ tg bdry

Genevieve Walsh IV

- Connections between hyp. and rel. hyp bdries
- Planar bdries of rel. hyp. grps ; conjectures
- More examples of cool boundaries

M compact, metrisable, perfect (no isolated pts) compactum.

[G acts on M as a convergence gp where all points are conical
$\iff$ G is hyperbolic and $\partial G \simeq M$]

(Bowditch - Tukia)

This goes through the space T of pairwise distinct triples of M.

$\rightarrow$ action is pD on T when G acts as a convergence group.

T/G cpt $\iff$ points of M are conical.

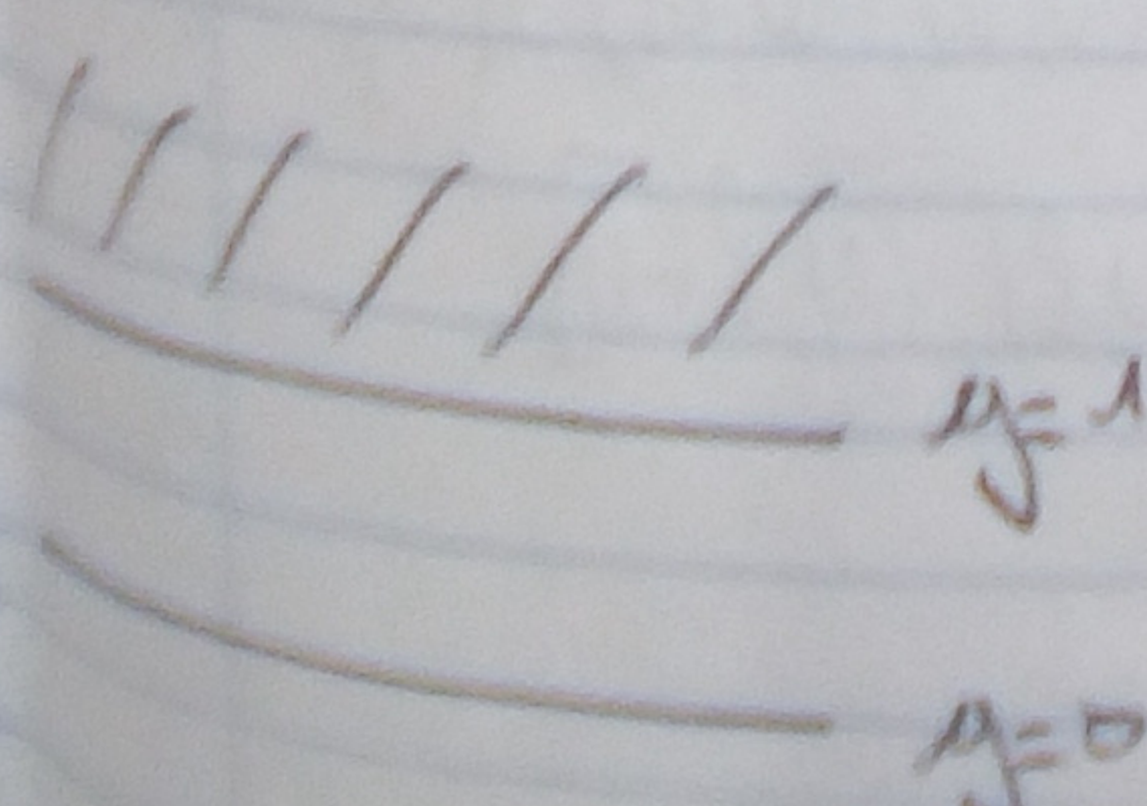
<u>Analog of this in rel. hyp. groups:</u>

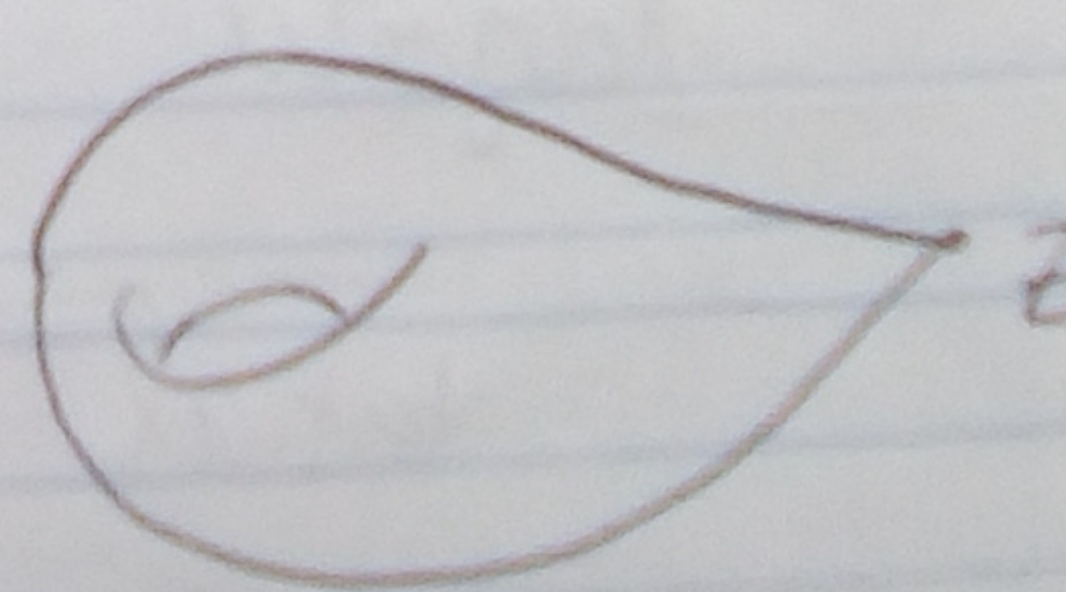
(Asli Yaman) M compact, metrisable perfect.

Γ acts on M as a GF convergence gp (all points are conical or bdd parabolic) $\iff$ (Γ, P) is rel. hyperbolic, and $\partial(\Gamma, P) = M$

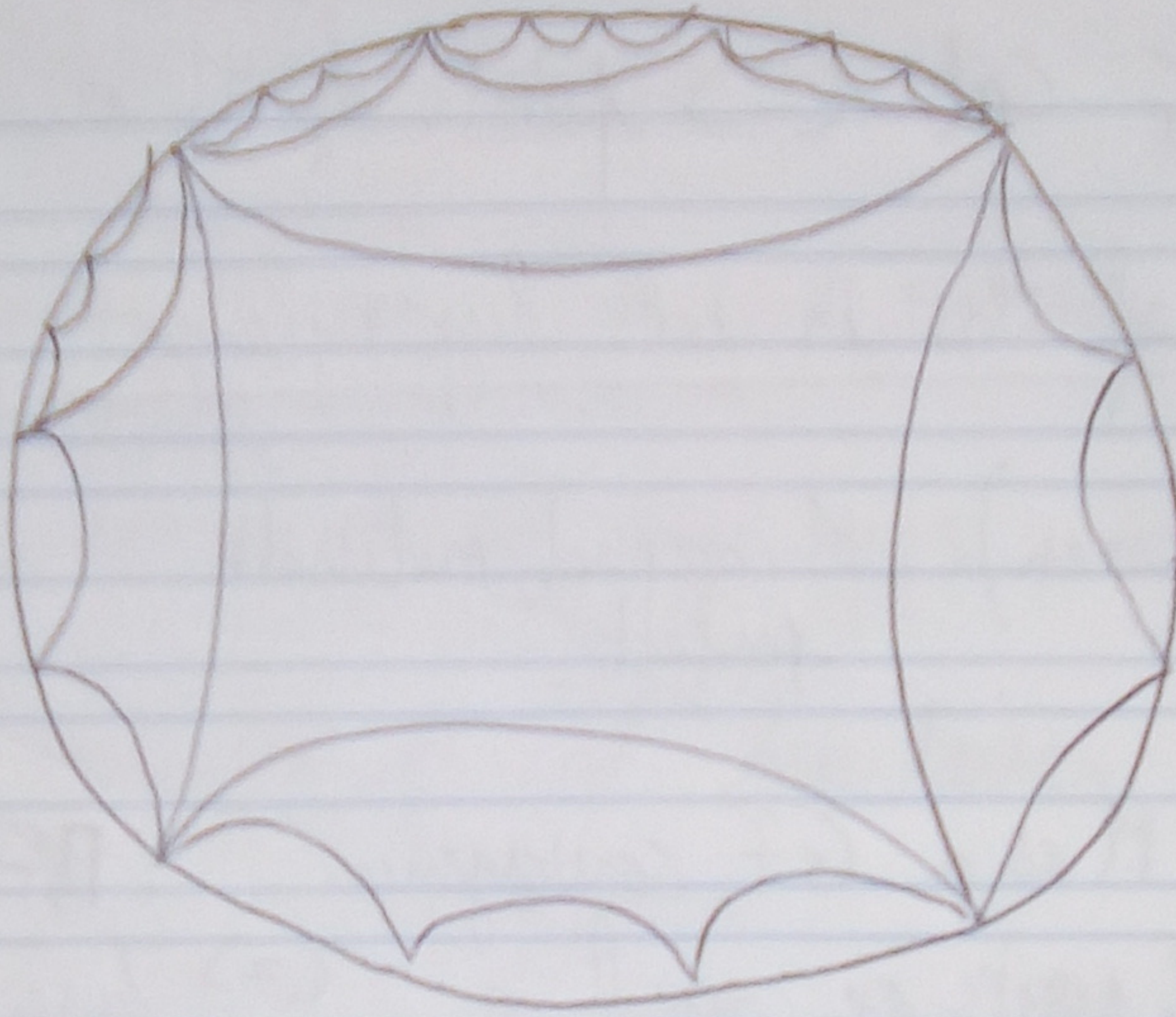
(and (G, P) acts nicely on T)

<u>Another way to get a space</u>





Bowditch, Cooper. (This version Groves–Manning, Sisto)



Γ graph

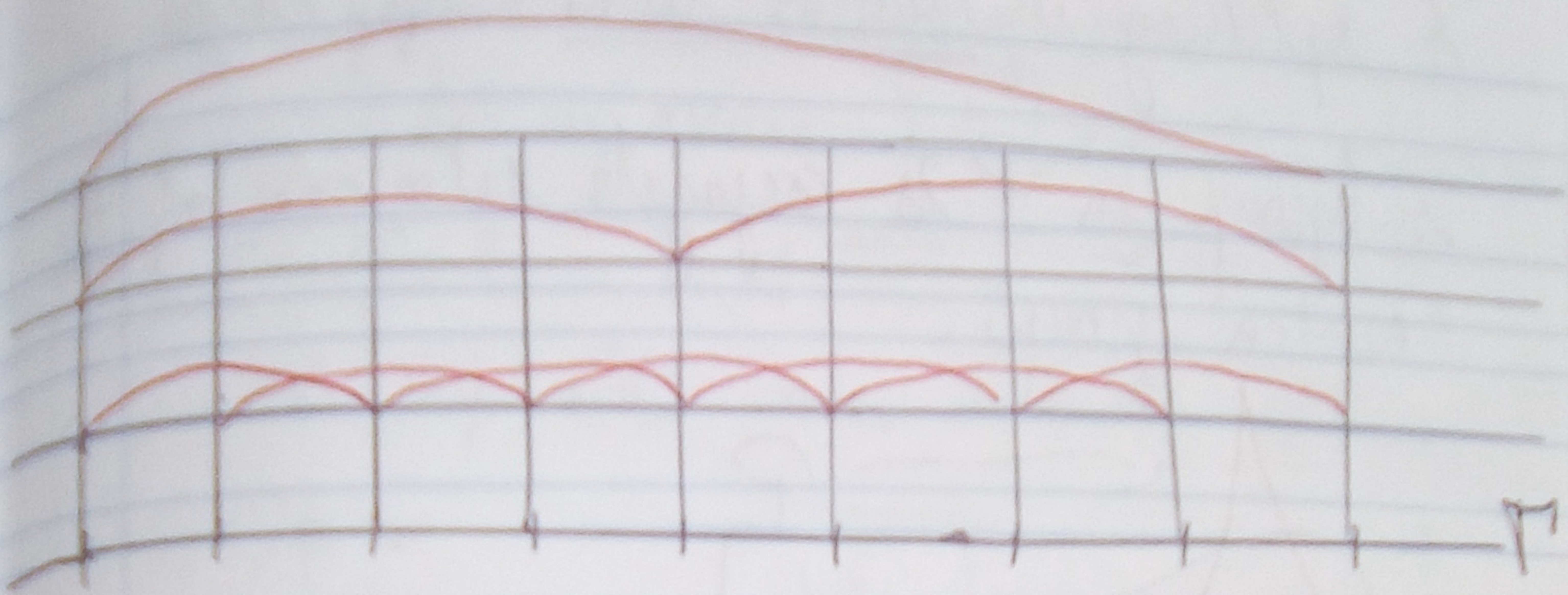
$\mathcal{H}(\Gamma)$: $V(\mathcal{H}(\Gamma)) = V(\Gamma) \times \mathbb{Z}_{\geq 0}$

$E(\mathcal{H}(\Gamma)) = [(v,k),(w,k)]$ if $(v,w) \in E\Gamma$

horizontal $\rightarrow$ $+ [(v,k),(w,k)]$ if $d_\Gamma(v,w) \leq 2^k$

vertical $\rightarrow$ $+ [(v,k),(v,k+1)]$

(Figure next page)



Cusped profile graph $X(G, \mathbb{P})$

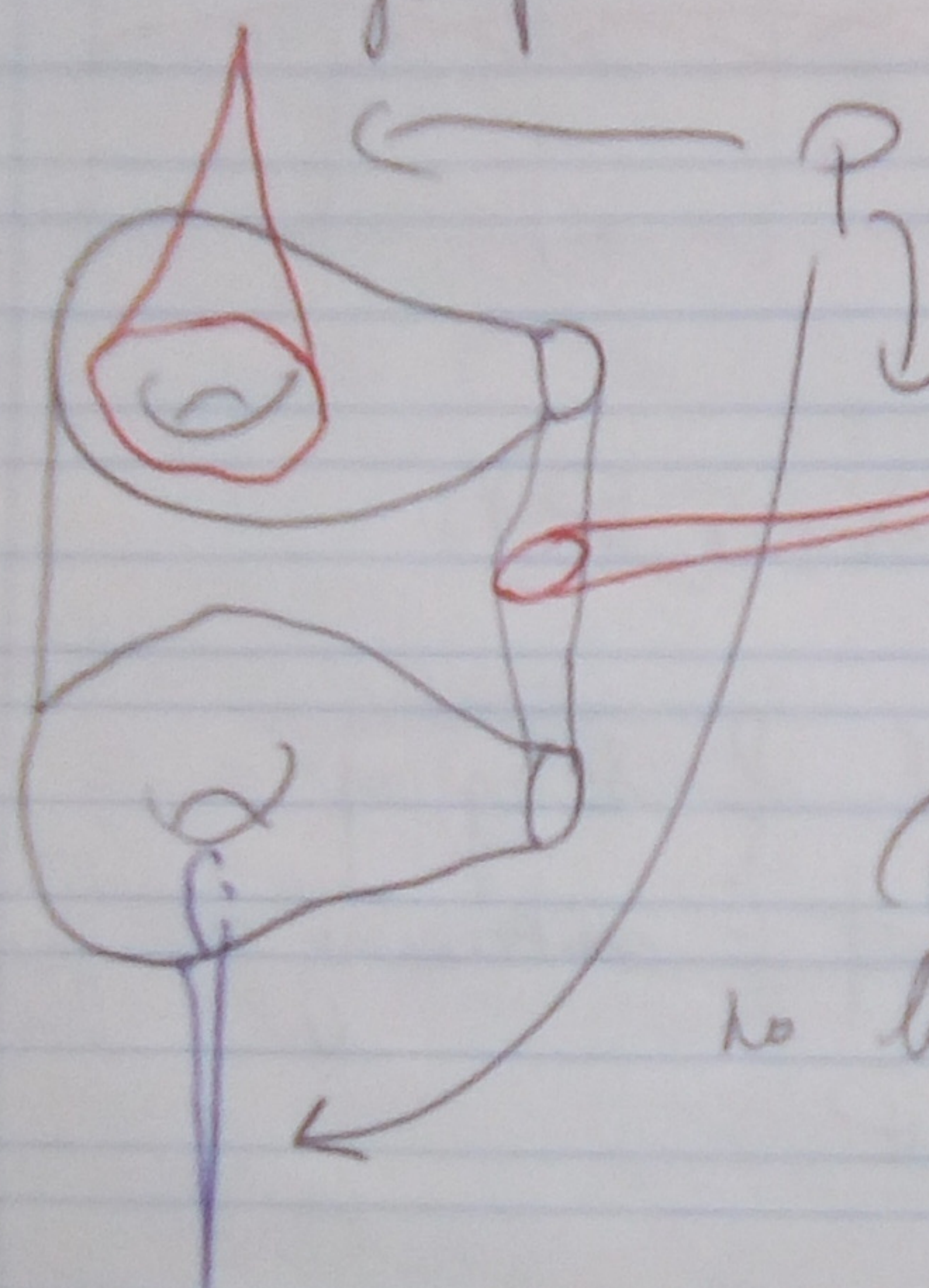
G group, $\mathbb{P}$ finite coll. of subgroups

$\mathcal{A}$: a generating set of G containing a generating set for each $P \in \mathbb{P}$

- $\mathcal{C}(G, \mathcal{A})$
- attach $\mathcal{H}(gP)$ to each coset gP

[Groves–Manning: (G, P) is relatively hyp. when $X(G, \mathbb{P})$ is hyperbolic, and $\partial X(G, \mathbb{P}) = \partial(G, P)$]

A splitting is <u>relative to P</u> if each subgroup in P is conjugate into one of the vertex groups.



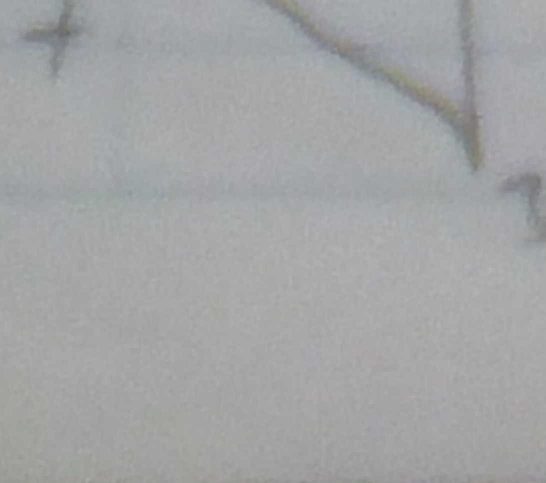
This does not have any splitting rel to P.

(need to check that no loop separates this space)

<u>Bowditch</u>: local cutpoint in ∂G $\Longleftrightarrow$ splittings over 2-ended gps

Except when $\partial G = S^1$ + group is rigid.

ex:



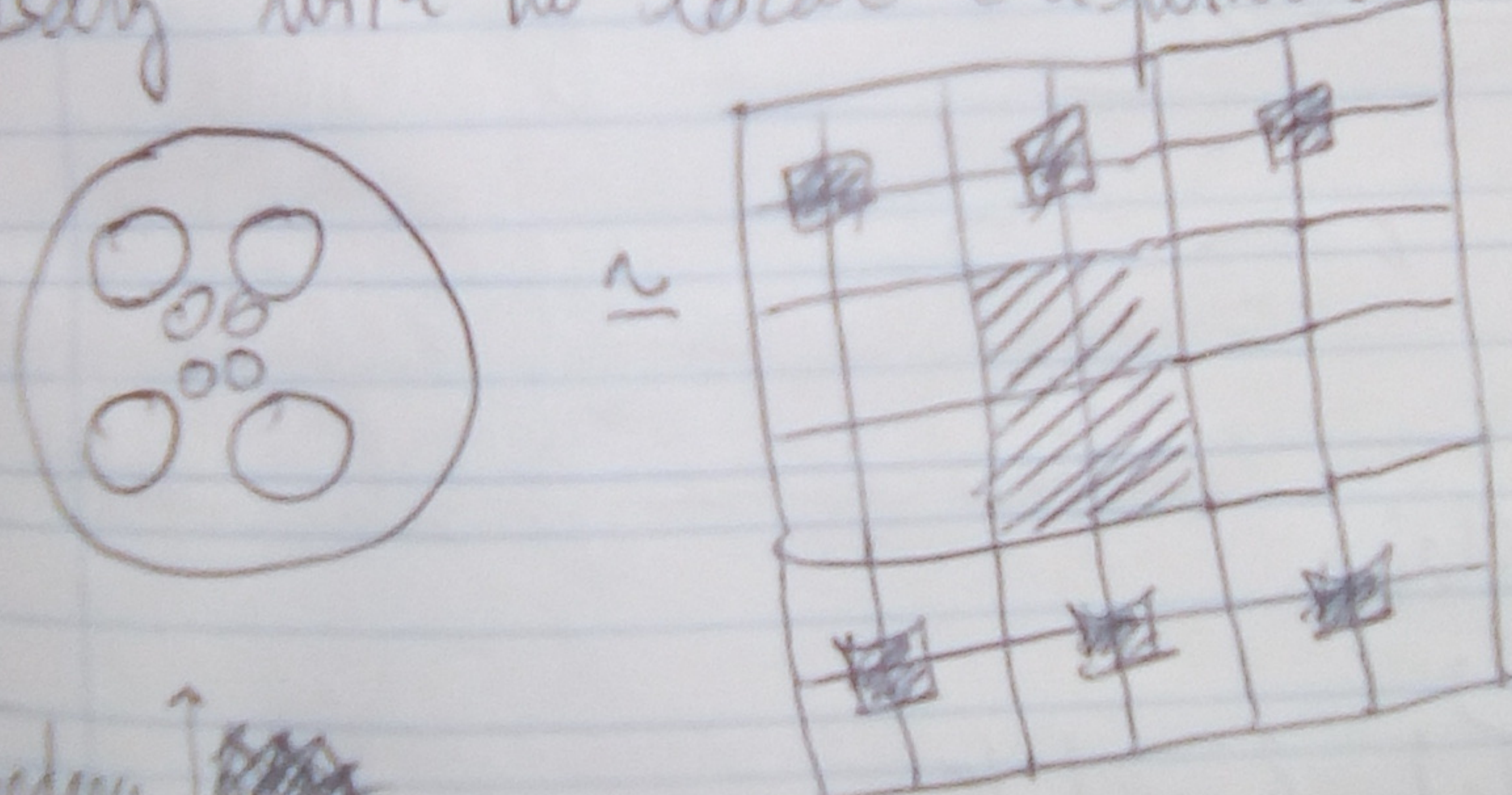
← doesn't have any splittings

(any hyp. triangle works)

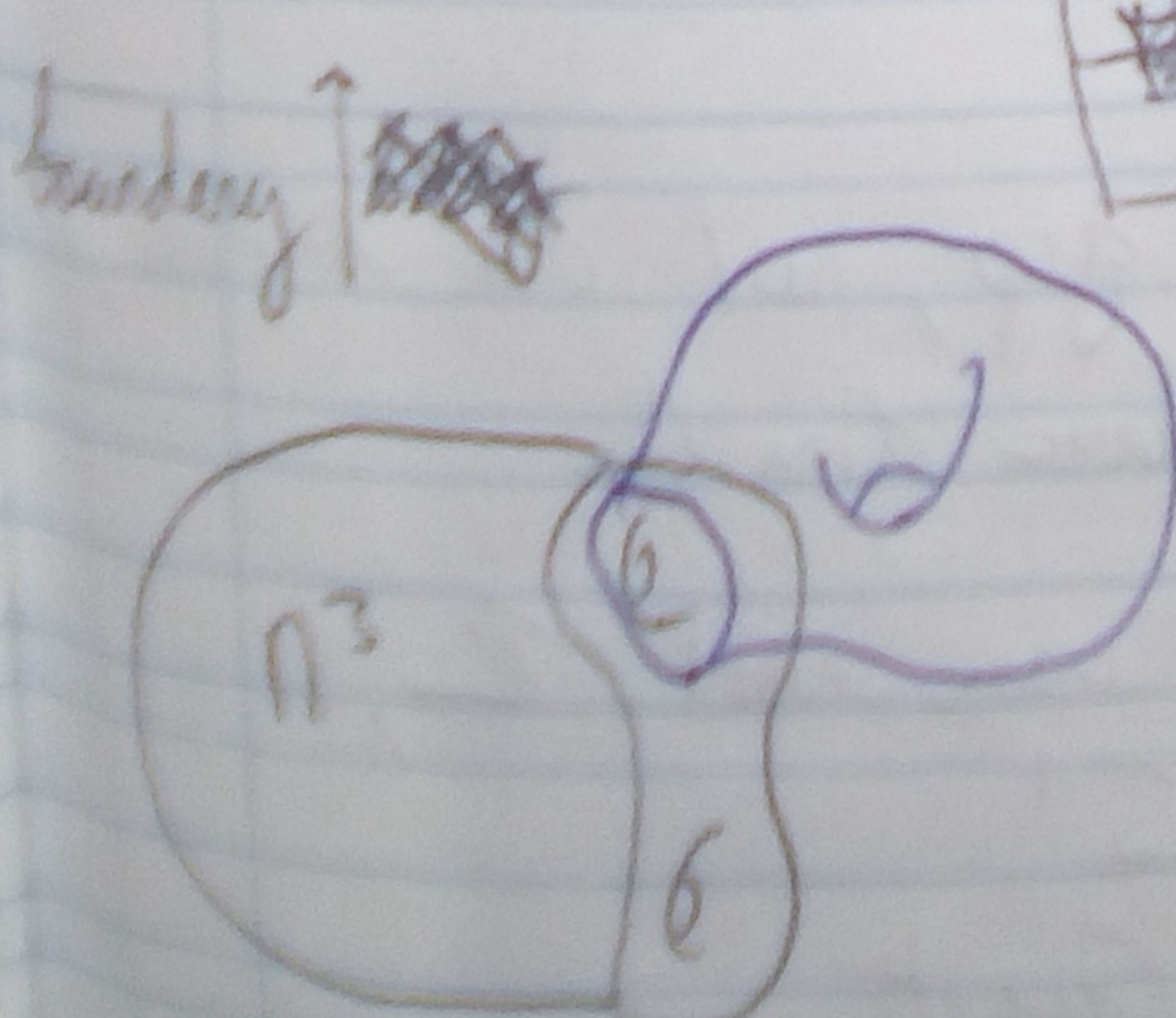
. Haulmark proved an analog of Bowditch's theorem (let us assume no cutpoint)

$$\left[\begin{array}{lcl} \text{local cutpoints} & & \text{splitting rel. } P \\ \text{in } \partial(G,P) & \Leftrightarrow & \text{over virtually cyclic} \\ \partial(G,P) \neq S^1 & & \text{groups} \end{array}\right]$$

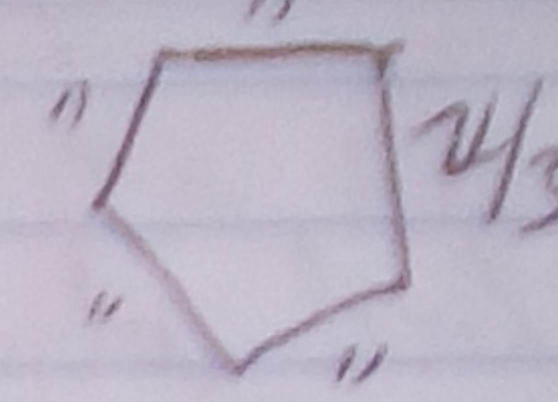
We need to understand the rigid piece.
Bdry with no local cutpoint.



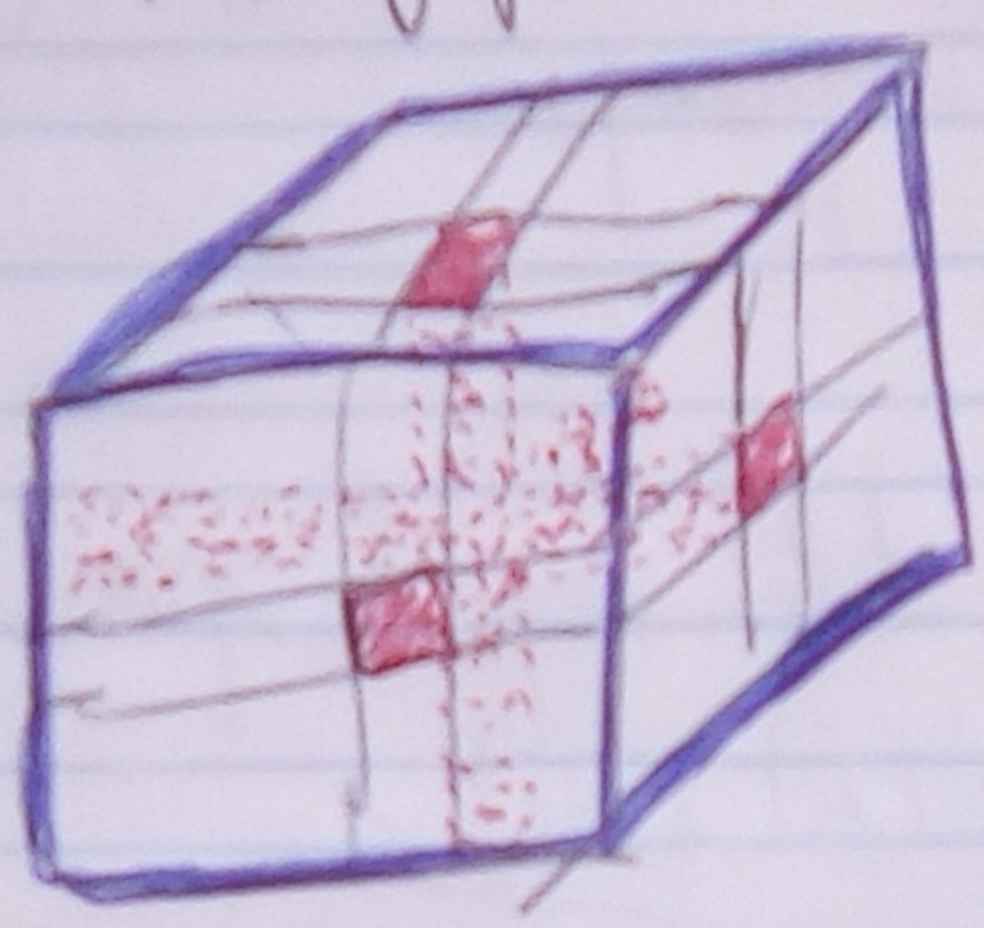
(Sierpinski) Carpet



Menger curve is the generic boundary

(ex: hyperbolic buildings) 

most hyp. grps have this as ∂G



remove each cube that do not touch the 1-squeletton

+ iterate infinitely many times

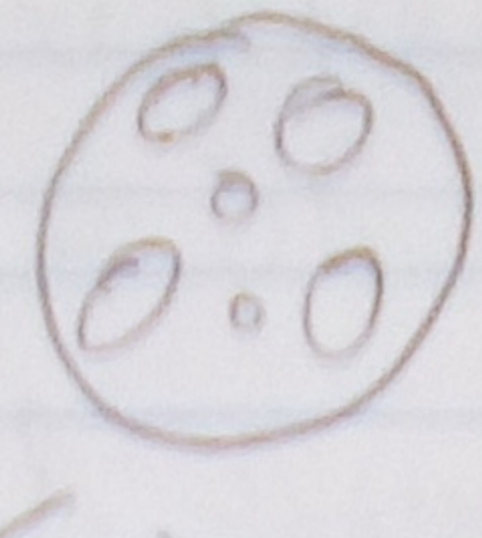
Kapovich-Kleiner

Let G 1 ended hyperbolic grp, w/ ∂G 1-dim, and G does not have dim 1.

If G doesn't split over virt. $\mathbb{Z}$, then ∂G:

1. S^1
2. Carpet
3. Menger curve

They conjecture:

If $\partial G \simeq$  (carpet)

Then, G is virtually $\pi_1(M^3)$

hyp. mfld w/ tg boundary

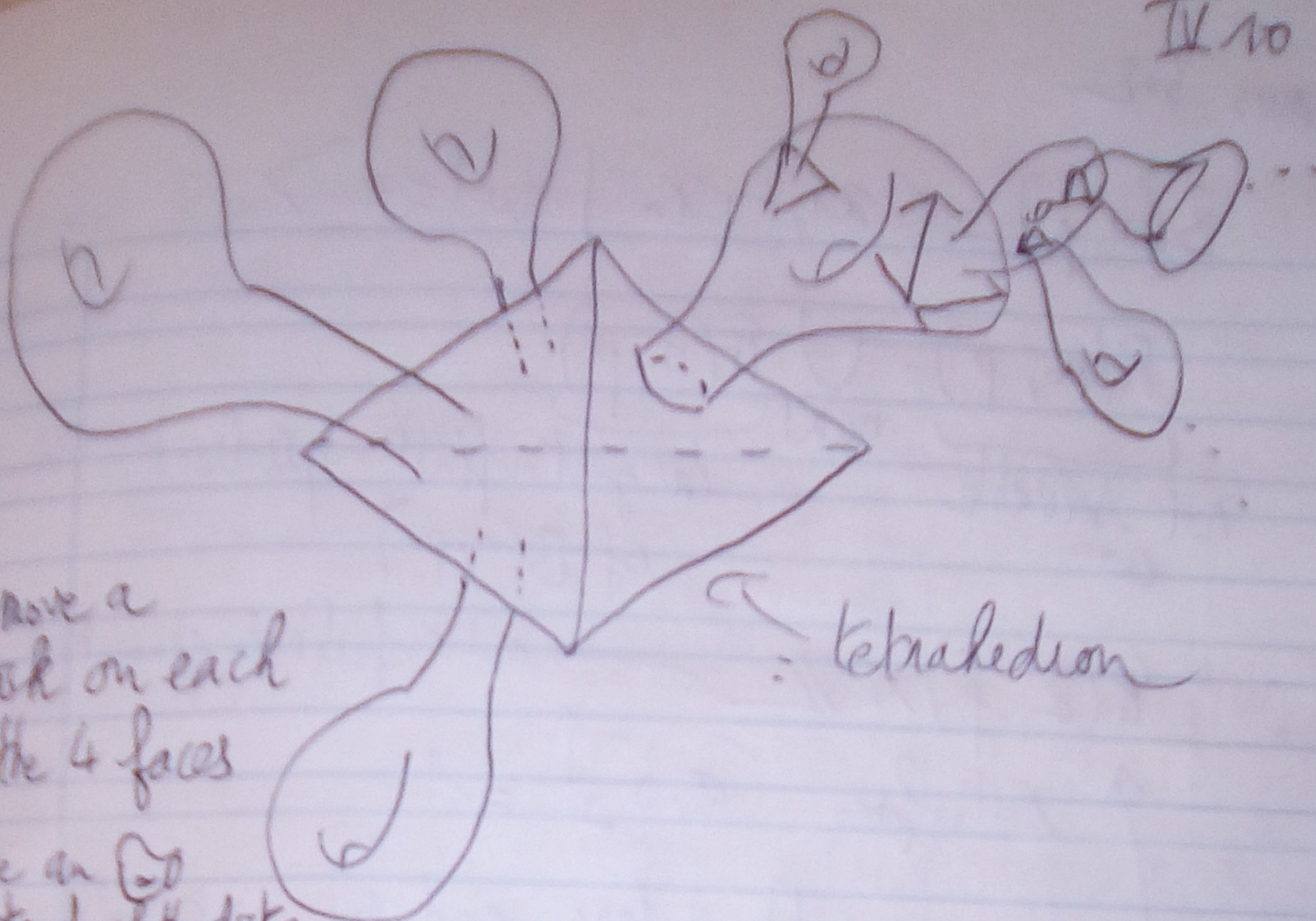
totally geodesic

Haulmark (Dasgupta) $(G, \mathcal{P})$ rel. hyp, $\dim \partial(G,\mathcal{P}) = 1$

$P \in \mathcal{P}$ one ended + G does not split rel $\mathcal{P}$ over virt. $\mathbb{Z}$.

Then, $\partial(G, \mathcal{P}) =$

1. S^1
2. Carpet
3. Menger curve



- remove a disk on each of the 4 faces
- glue an instead of the disks
- triangulate the torri and glue other torri, ... (reiterate ∞ many times)

This is the bdary of a RACG (whose flagification is a torus)

Q: which groups have this as a boundary?