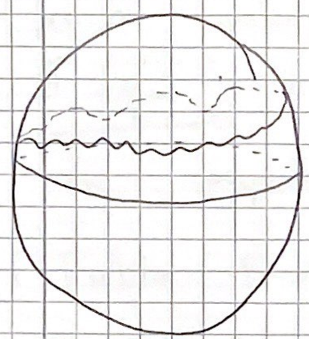
Lecture 3. Genevieve Walsh

So far, the examples of ∂G have been Kleinian

Kleinian: discrete subgroup of $PSL(2,\mathbb{C}) = Isom^+(\mathbb{H}^3)$

For nice groups H of $Isom^+(\mathbb{H}^3)$, the ∂H is

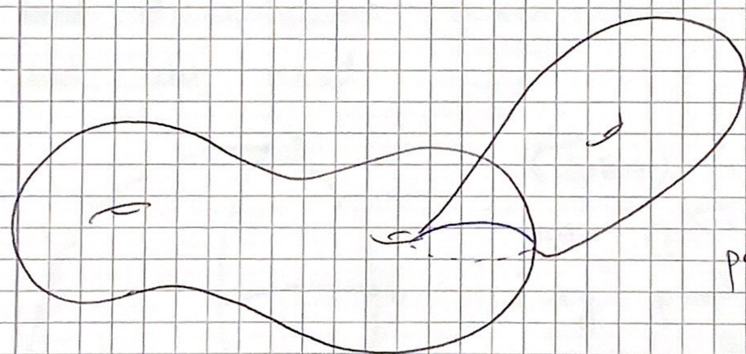
ΛH = orbit(x) under $H \cap \partial \mathbb{H}^3 = S^2$



For Kleinian gps, the ∂ is planar, because ∂H embeds into S^2

planar: embeds into S^2

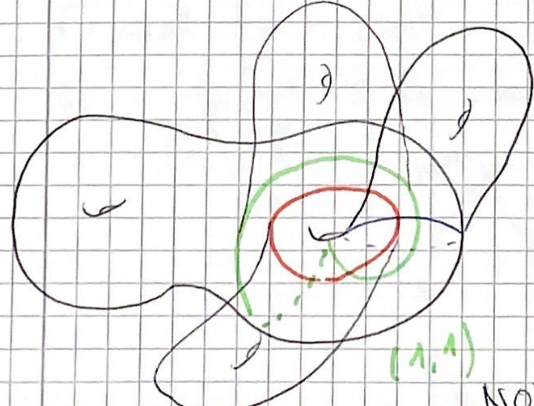
Let's look at a group whose boundary is not planar

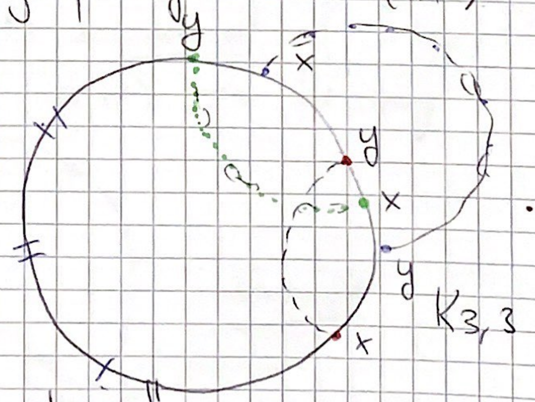


Thicken up parts, glue along boundary

not a surface, but can be realised as a 3-manifold

Hyperbolic 3-manifold w/ boundary. Thurston - this is realised as a group of $Isom(\mathbb{H}^3)$





NOT planar!!

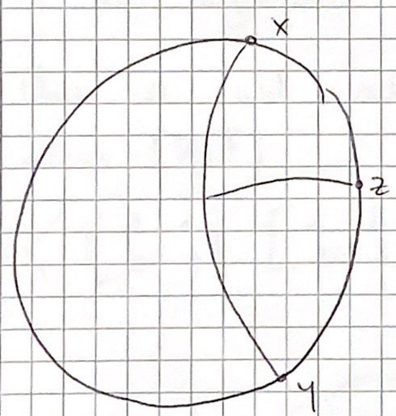
(2) Discrete subgroups of $PSL(2,\mathbb{C})$ are cool in that they are convergence group. (Gehring + Martin)

<u>Defn</u> (discrete) convergence group. M cpt Hausdorff space. A group of homeomorphisms of M is a convergence group if (g_i) distinct $\Rightarrow$ $\exists\ (g_{n_i}), a, b$ such that $g_{n_i} \to a$ uniformly on compact subsets of $M \setminus \{b\}$

pts on M

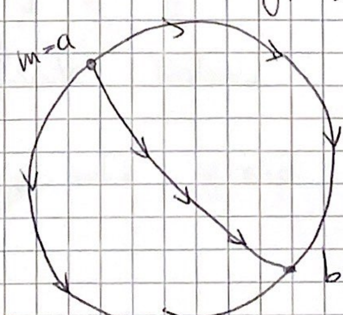
(Tukia, Bowditch) A hyperbolic group acts as a convergence group on its boundary.

A convergence group action on $M \Rightarrow$ properly ~~dist~~ discontinuous action on $T = (x,y,z) \in M^3$, not all equal



(Tukia) A convergence group has $G \setminus T$ compact $\Leftrightarrow$ every point of M is a conical limit point for the action of G.

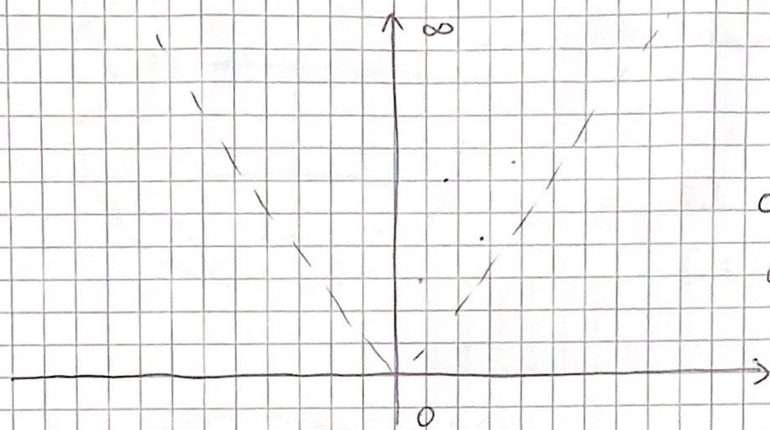
Conical limit point : $m \in M$ is <u>conical</u> if $\exists (g_i) \subset G$ st $g_i(m) \to a$ but $g_i(x) \to b$ for any $x \in M \setminus \{m\}$



<u>Example</u>

③ $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$ $\quad z \longmapsto \dfrac{az+b}{cz+d}$

$z \longmapsto 2z$



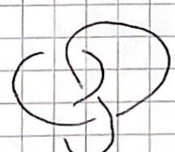
0 is a conical limit pt.

<u>Bowditch</u> $G \curvearrowright M$ as a convergence gp is hyperbolic iff every point of M is a conical limit point.

G is hyperbolic $+ \partial G = M$

This is not all that happens dynamically on $\mathrm{Isom}(\mathbb{H}^3)$

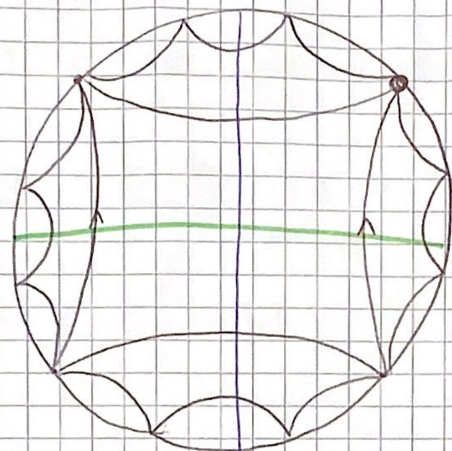
$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \right\rangle$ $\quad \infty$ is not a conical limit point.

$\pi_1 (S^3 \setminus$  $)$ hyperbolic knot complement

There is a group of $\mathrm{Isom}^+(\mathbb{H}^3)$, Γ, where $\mathbb{H}^3 / \Gamma$ has finite volume.

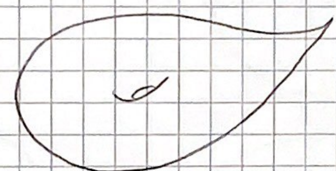
<u>NOT</u> a hyperbolic group. $\mathbb{Z} \oplus \mathbb{Z} \not\leq$ hyp gp!

(4) Example



$[a,b]$ is parabolic
- fixes one point on the boundary

This is a relatively hyperbolic gp pair
$(\mathbb{F}_2, \langle [a,b] \rangle)$

π_1 () $= \mathbb{F}_2$

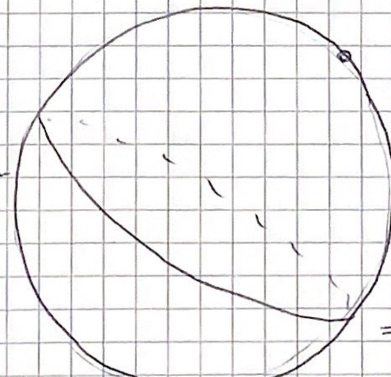
Relatively hyperbolic group pair (Bowditch, Farb)
$(G, \mathcal{P})$ is relatively hyperbolic if G acts properly discont & by isom on X : proper hyperbolic space st every pt of ∂X is either (a) conical limit point
(b) bounded parabolic points

m is parabolic

$\exists$ parabolic subgroup fixing m, P_m + $\left(\partial X \setminus \{m\}\right) / P_m$

is compact

+ $\mathcal{P}$ is exactly the collection of maximal parabolic subgroups



stabilised by $\mathbb{Z} \oplus \mathbb{Z}$.

$S^2 \setminus pt / \mathbb{Z} \oplus \mathbb{Z}$

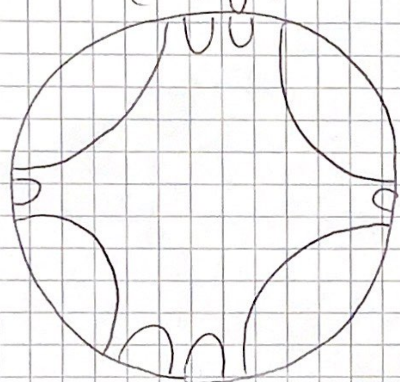
$= \text{plane} / \mathbb{Z} \oplus \mathbb{Z} = T^2$.

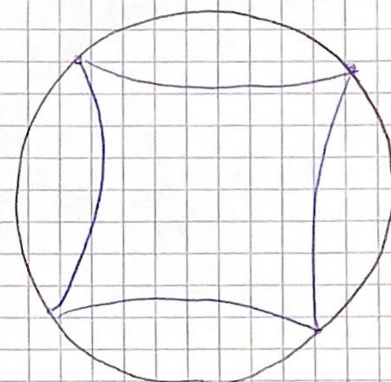
(5) $\partial(G, \mathcal{P}) = \partial X$

- It is not, in general, true that if $(G, \mathcal{P}) \underset{\text{rel hyp}}{\curvearrowright} X$ and $(G, \mathcal{P}) \underset{\text{rel hyp}}{\curvearrowright} Y$ then X qi Y.

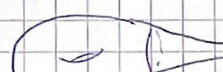
But (Bowditch) $\partial X \cong \partial Y$

So this boundary is well-defined for the pair $(G, \mathcal{P})$

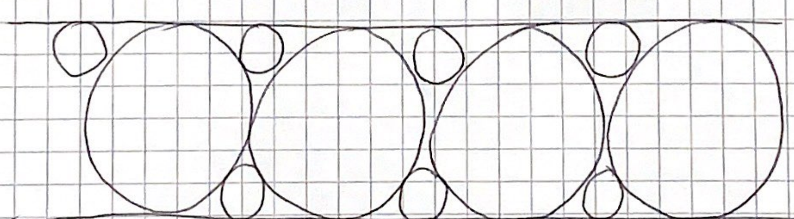




$\partial(\mathbb{F}_2, \emptyset) = \mathcal{C}$



$\partial(\mathbb{F}_2, \langle [a,b] \rangle) = S^1$



$\partial(\mathbb{F}_2, \langle [a,b] \rangle, \langle a \rangle, \langle b \rangle)$