Geneviève Walsh III

So far, the examples of ∂G are Kleinian

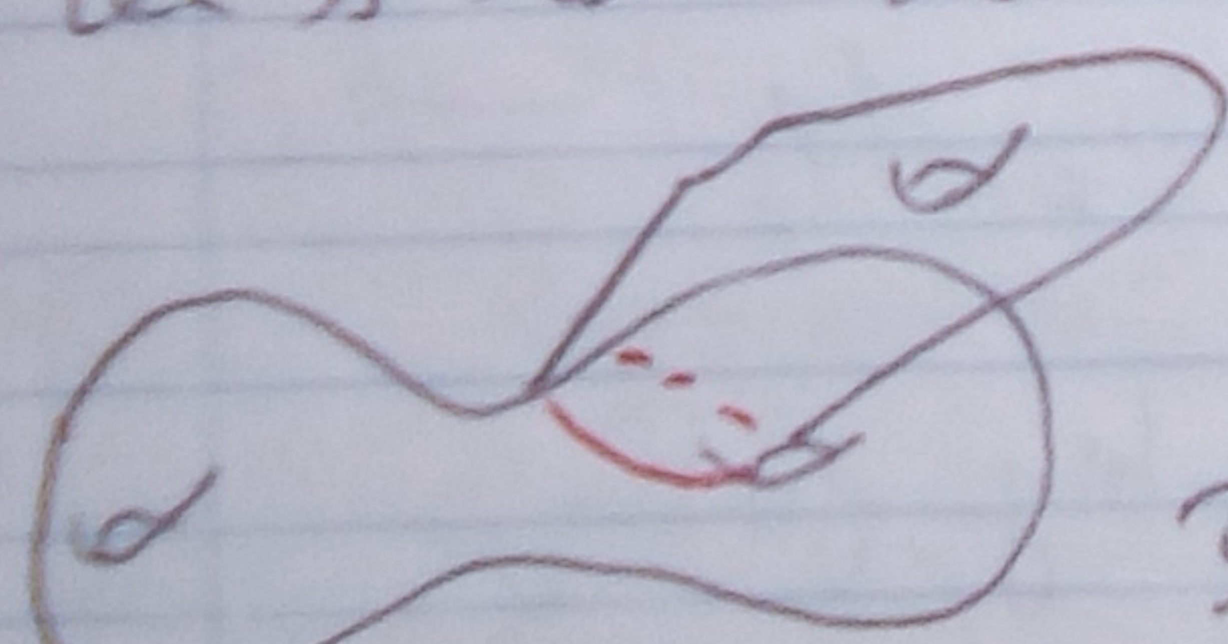
Kleinian : discrete subgroup of $PSL_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$

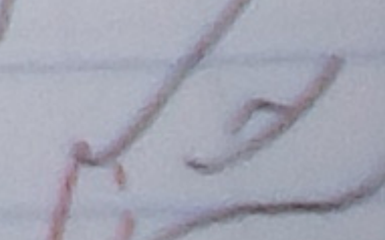
For nice groups $H < \mathrm{Isom}^+(\mathbb{H}^3)$,

$$\partial H = \Lambda H = \overline{\text{orbit}(x) \text{ under } H} \cap \partial\mathbb{H}^3 = S^2$$

For Kleinian group, ∂ is planar (b/c $\partial H \hookrightarrow S^2$)

Let's look at a group whose bdry is not planar



can be realized as a 3 mnfld (w/ boundary, needs to thicken the additional  and maybe the rest also)

→ From Thurston's geometrization, this is realized a group of $\mathrm{Isom}(\mathbb{H}^3)$

The boundary is planar

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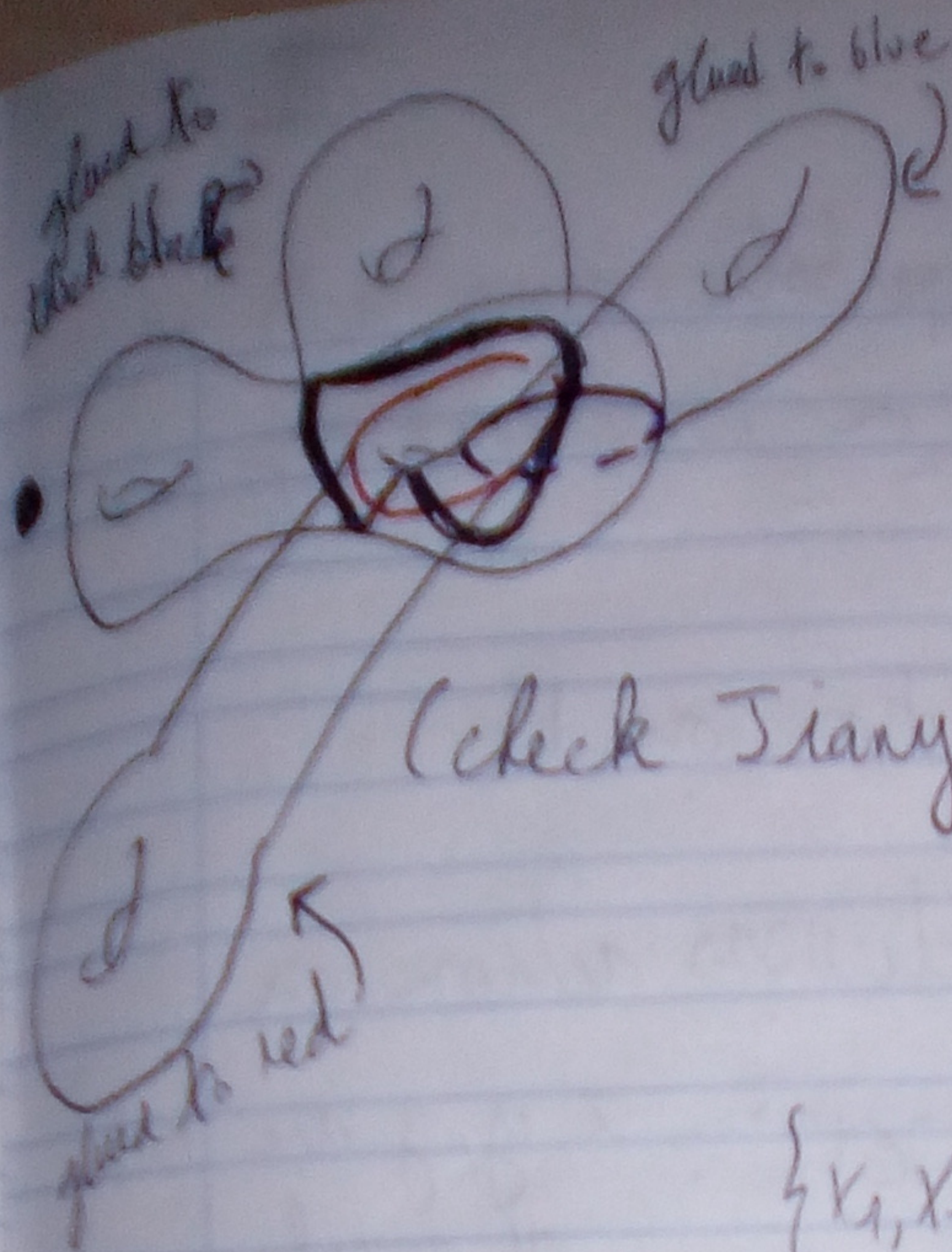
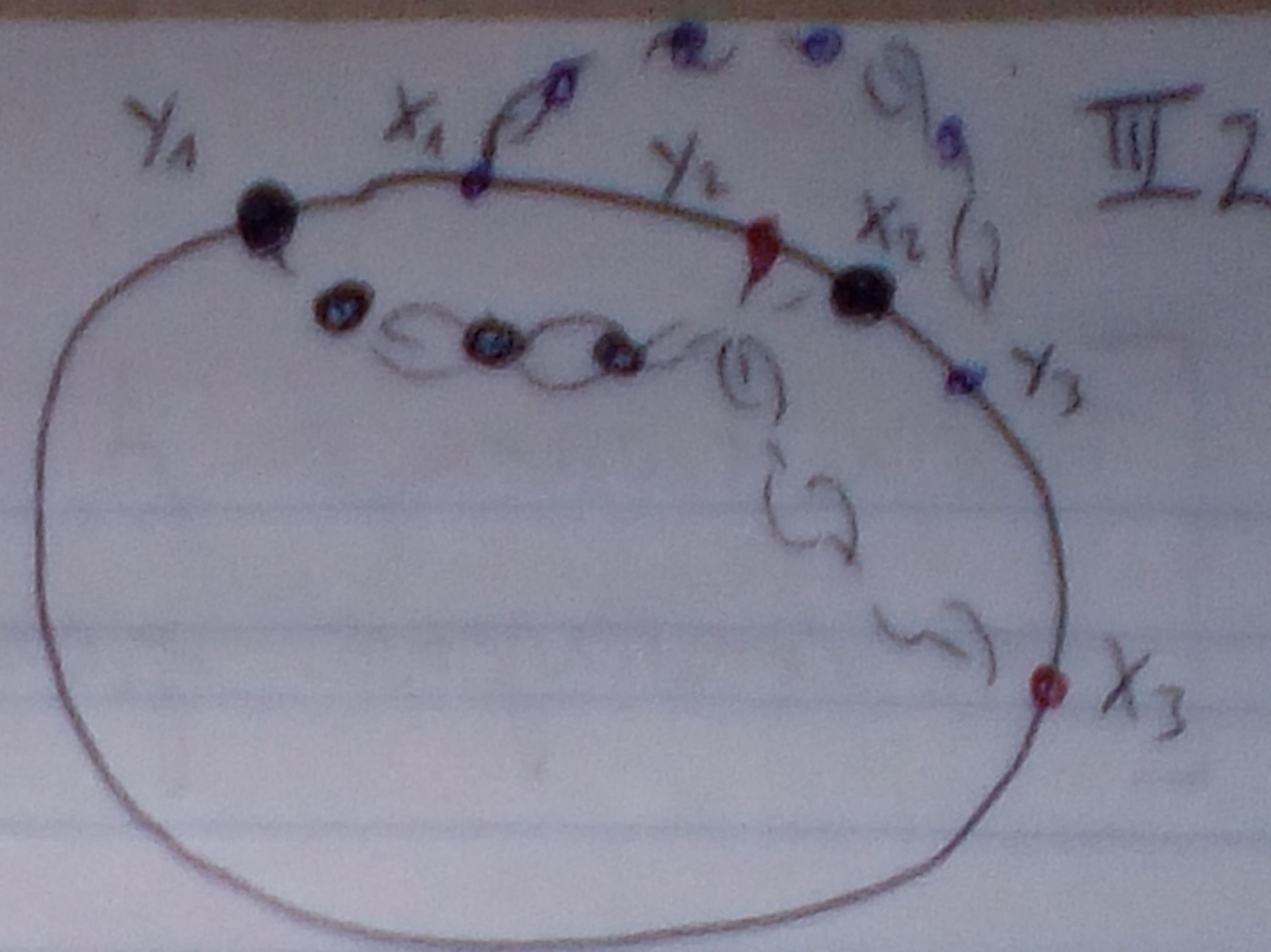




(check Jianyi Lou on arXiv)

$\{x_1, x_2, x_3\} \sqcup \{y_1, y_2, y_3\} \simeq K_{3,3}$;

so the boundary is <u>not planar</u>.

Discrete subgroups of $PSL_2(\mathbb{C})$ are "convergence groups" (Gering + Martin)

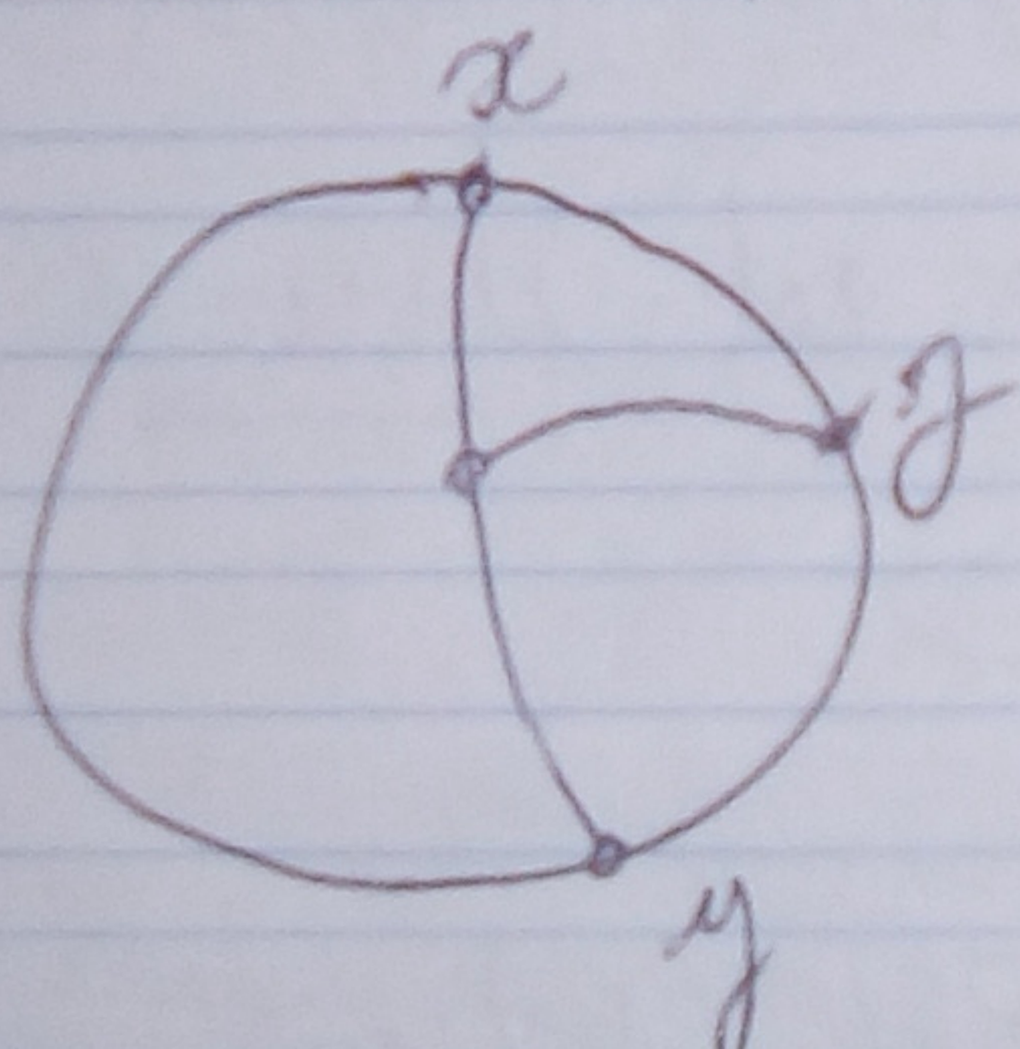
<u>Definition</u>: (discrete) convergence group.
let $G < \mathrm{Homeo}(M)$, M cpt Hausdorff
G is a convergence group if
(g_i) distinct $\Rightarrow \exists (g_{n_i}), (a,b) \in M^2$: $g_{n_i} \longrightarrow a$ uniformly on compact sets of $M \setminus \{b\}$

[Tukia, Fredon: A hyperbolic group acts as a convergence group on its boundary

A convergence group action on M

$\Rightarrow$ properly discontinuous action on

$T = \{(x,y,z) \in M^3 : x, y, z \text{ are not all equal}\}$

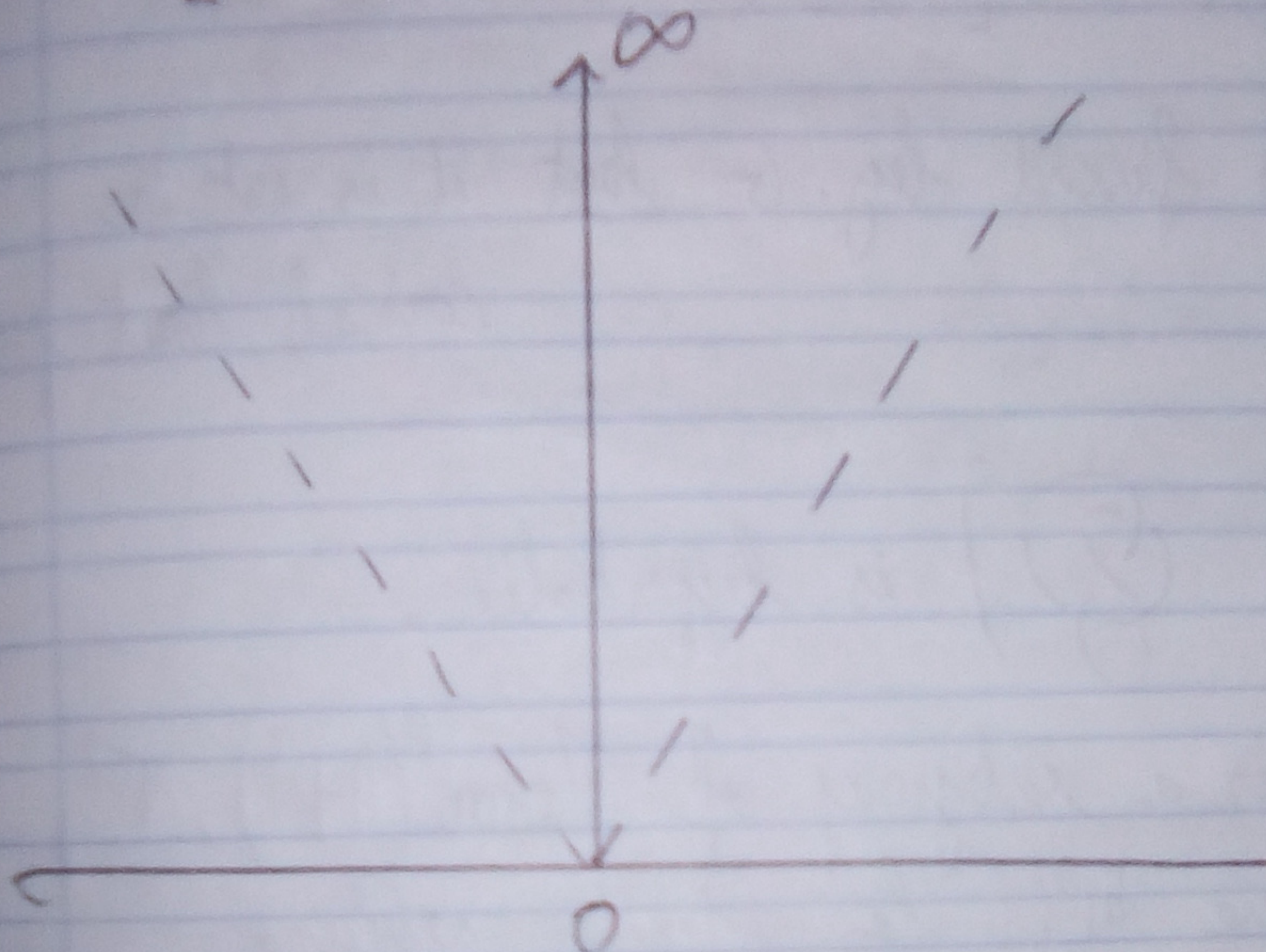


[Tukia: a convergence group has T/G compact $\Leftrightarrow$ every point of M is a conical limit point for the action of G

Conical limit point: $m \in M$ is <u>conical</u> if

$$\exists\, g_i \in G,\ a \neq b : \begin{cases} g_i(m) \to a \\ g_i(x) \to b \text{ for any other } x \in M. \end{cases}$$

ex: $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{bmatrix} \curvearrowright \mathbb{H}^2$ (multiplication by 2)



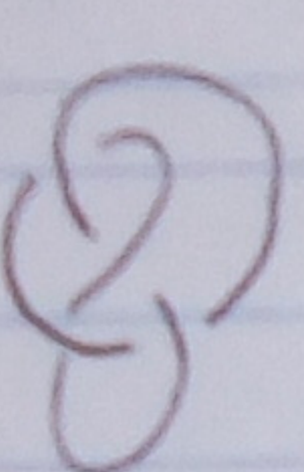
Bowditch: $G \curvearrowright M$ acting as a convergence gp
G is hyperbolic $\Leftrightarrow$ every point of M is a conical limit point

(In that case, $\partial G = M$)

This is not all that happens dynamically on $Isom(\mathbb{H}^3)$

$G = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \right\rangle \simeq \mathbb{Z}^2$

∞ is fixed by G <u>but</u> it is not a conical limit point

$\pi_1 (S^3 \setminus$  $)$ is hyperbolic.

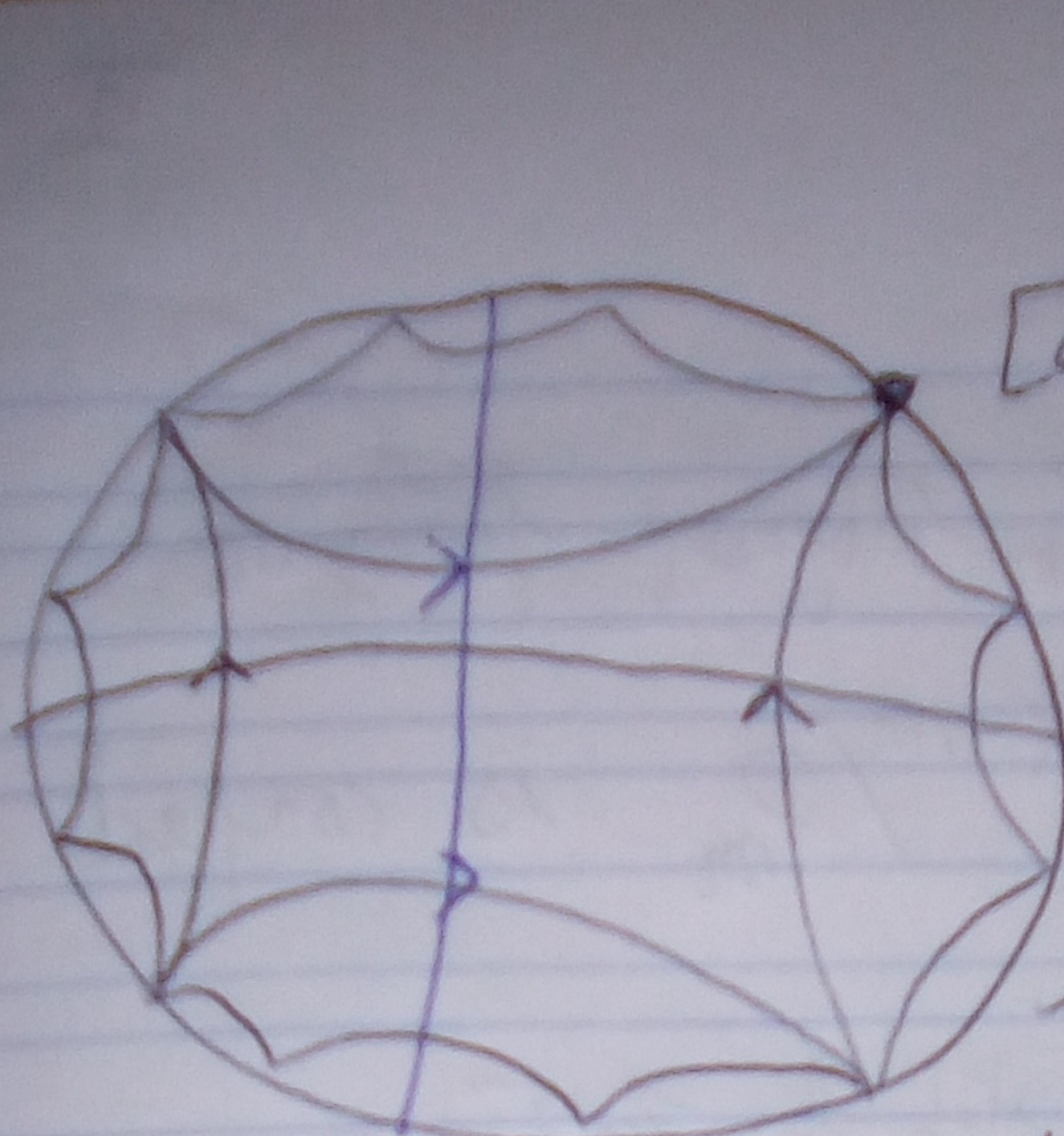
There is a subgroup of $Isom^+(\mathbb{H}^3)$, Γ, where $\mathbb{H}^3/\Gamma$ has finite volume

ex of a non hyperbolic group:

$\mathbb{Z} \oplus \mathbb{Z}$ $\not<$ hyperbolic

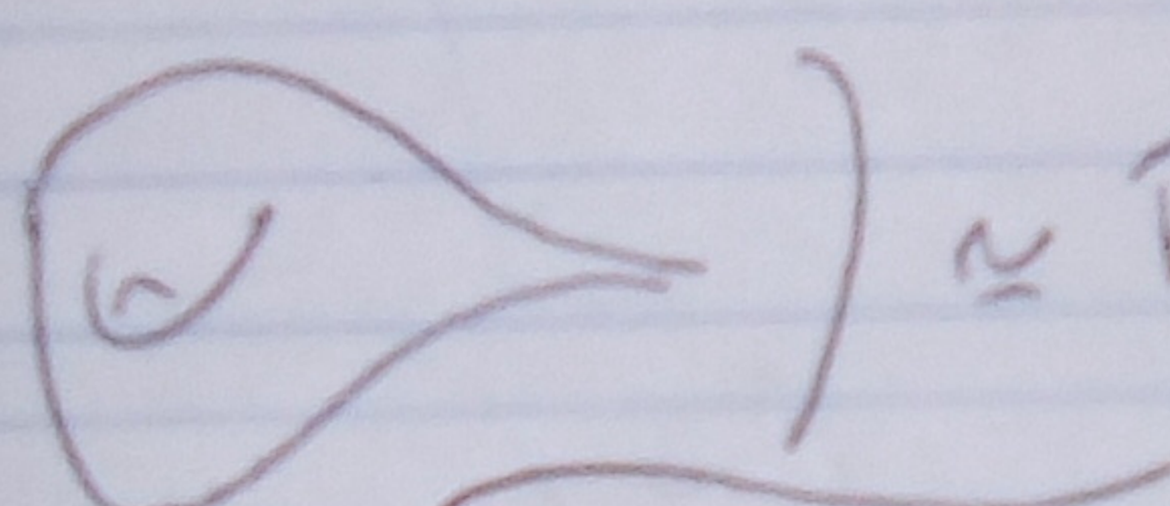
(see next page)

Ex:



$[a,b]$ is parabolic (fixes one point on the boundary)

This is a relatively hyperbolic group pair $(F_2, \langle [a,b] \rangle)$

$\pi_1\Big($  $\Big) \simeq F_2$

Relatively hyperbolic group pair (Bowditch-Farb)

(G,P) rel. hyp if:

G acts properly discontinuously, by isometries on X (proper hyperbolic space)

s.t. every point of ∂X is either:

(a) conical limit point

(b) bounded parabolic point

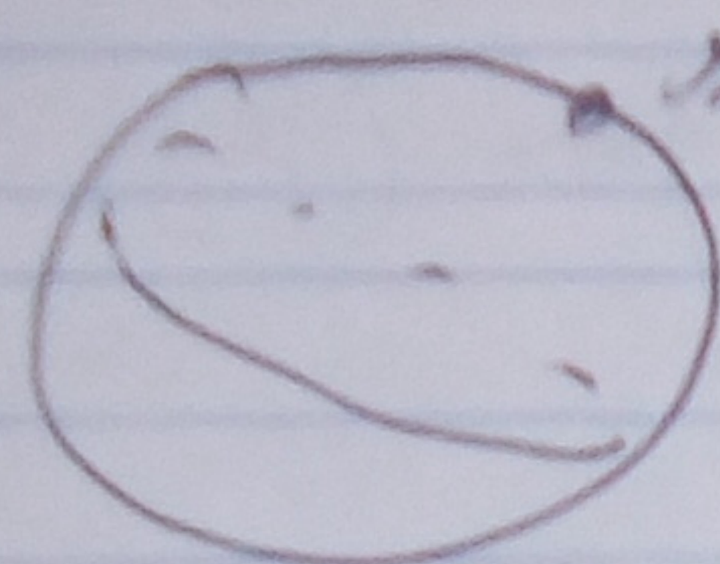
(see next page)

m is parabolic:

$\exists$ parabolic P_m subgroup fixing m

s.t. $\partial X - m / P_m$ is compact

recall:



$$S^2 \setminus pt / \mathbb{Z}\oplus\mathbb{Z} \simeq \text{plane} / \mathbb{Z}\oplus\mathbb{Z} = \underbrace{\text{torus}}_{\text{compact}}$$:)

$\boxed{\partial(G,P) := \partial X}$ (this is a definition)

NB: it is not true in general that:

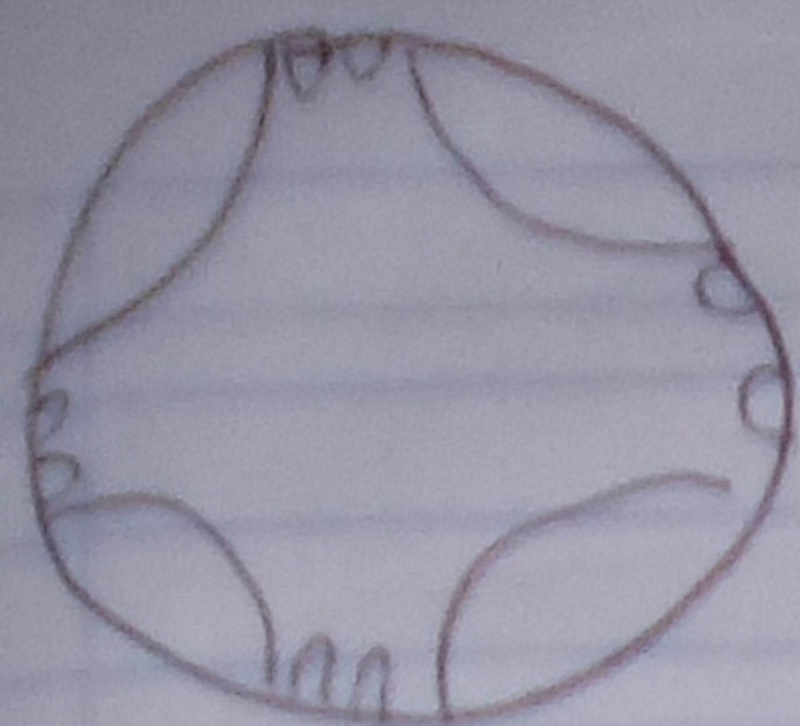
$$(G,P) \underset{\text{rel. hyp}}{\curvearrowright} X \text{ and } (G,P) \underset{\text{rel. hyp}}{\curvearrowright} Y$$

then $X \overset{q.i}{\simeq} Y$

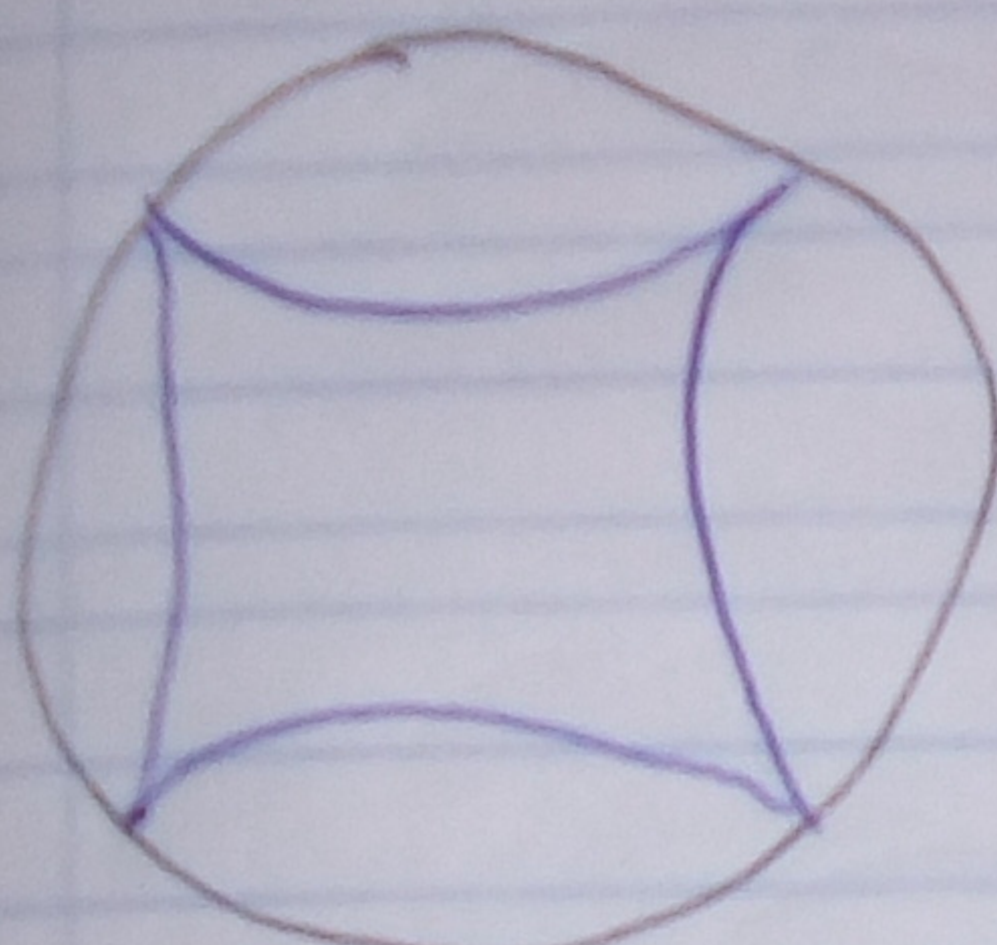
But (Bowditch): $\partial X \cong \partial Y$

so $\partial(G,P)$ is well-defined.

Few examples :

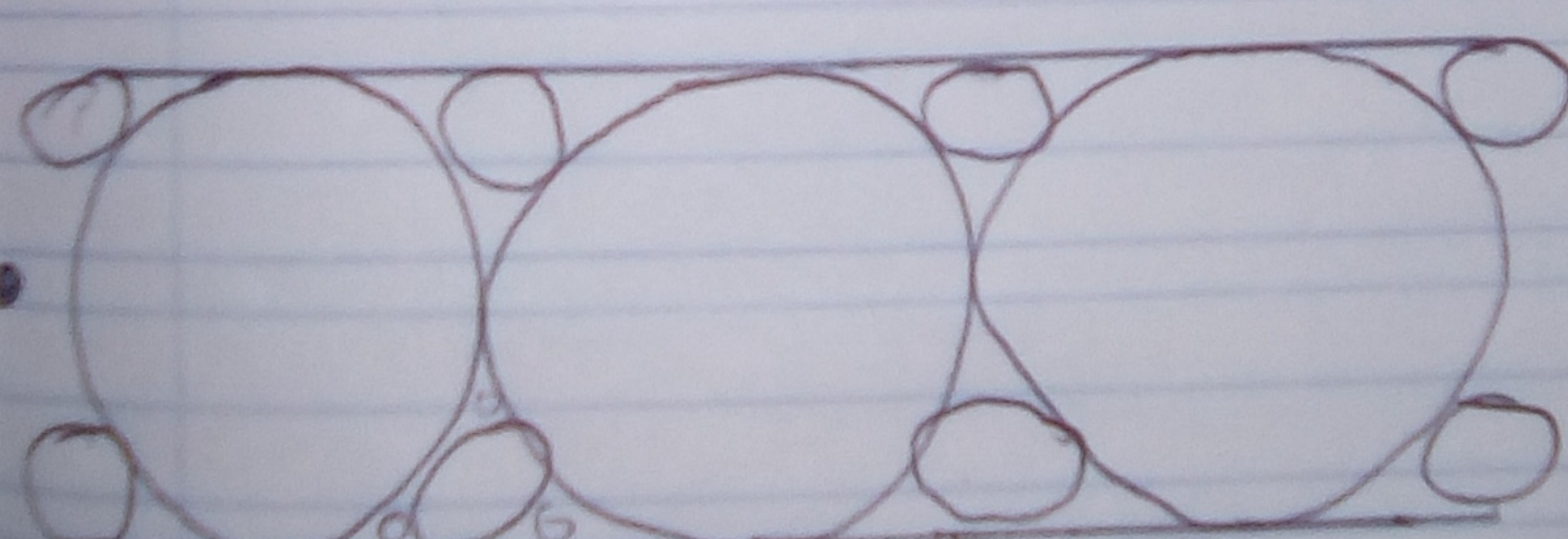
- 

  $\partial(F_2, \emptyset) = \mathcal{C}$ (Cantor set)

- 

  $\partial(F_2, \langle a,b \rangle) = S^1$

(you crush the limit set of the parabolic subgroup to a point)

- 

↳ $= \partial(F_2, \langle [a,b] \rangle, \langle a \rangle, \langle b \rangle)$