Genevieve Welsh II

Topology on the boundary

X proper geodesic hyp. metric space

$x \in \partial X \quad N_r(x) = \{ y \mid (x,y)_w > r \}$

$$(x,y)_w = \sup_{\substack{x_i \to x \\ y_i \to y}} \{ \liminf_i (x_i, y_i)_w \}, \quad y \in \partial X$$

$$(x,y)_w = \sup_{x_i \to x} \{ \liminf_i (x_i, y)_w \}, \quad y \in X$$

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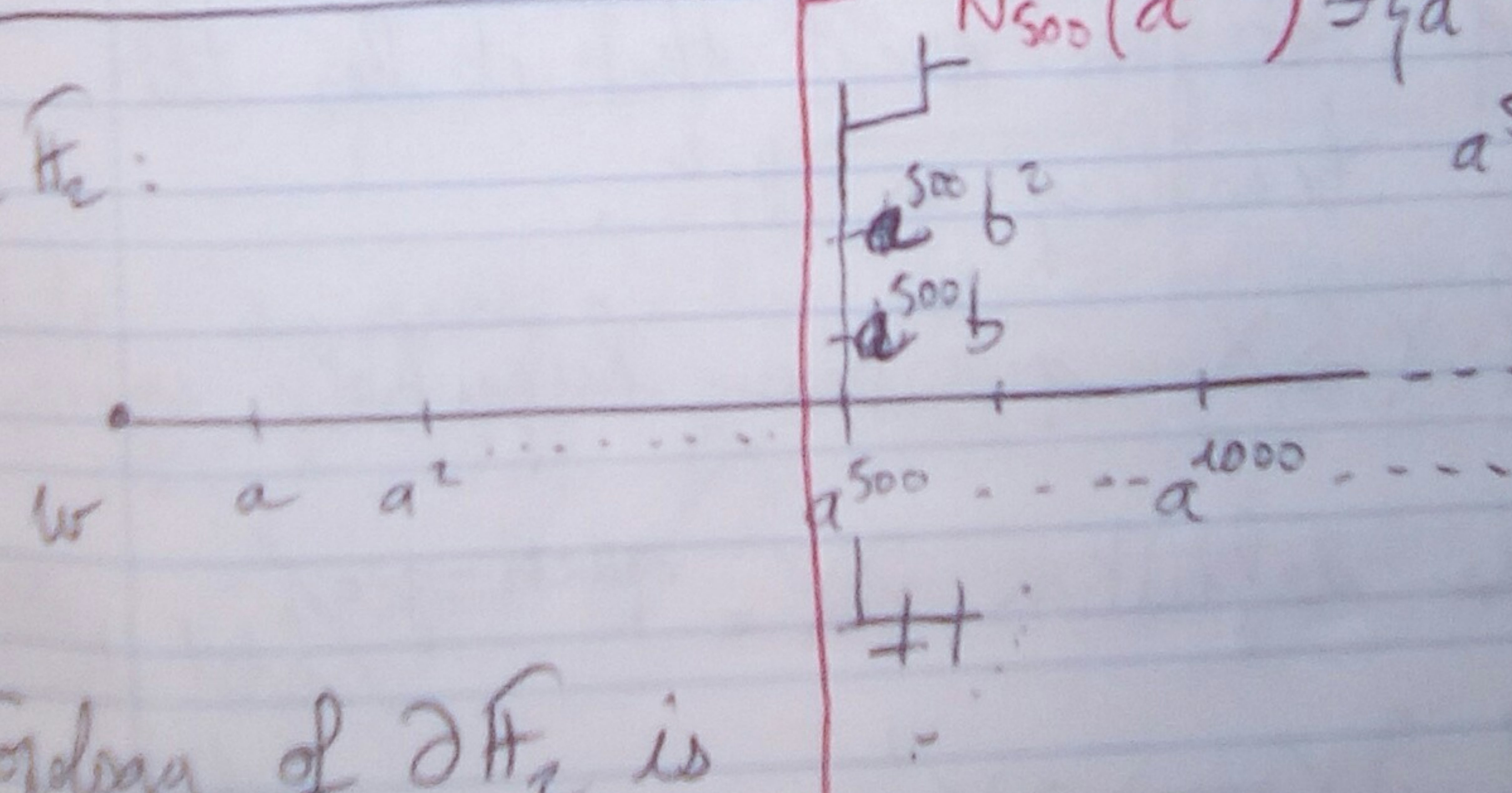
in F_2:

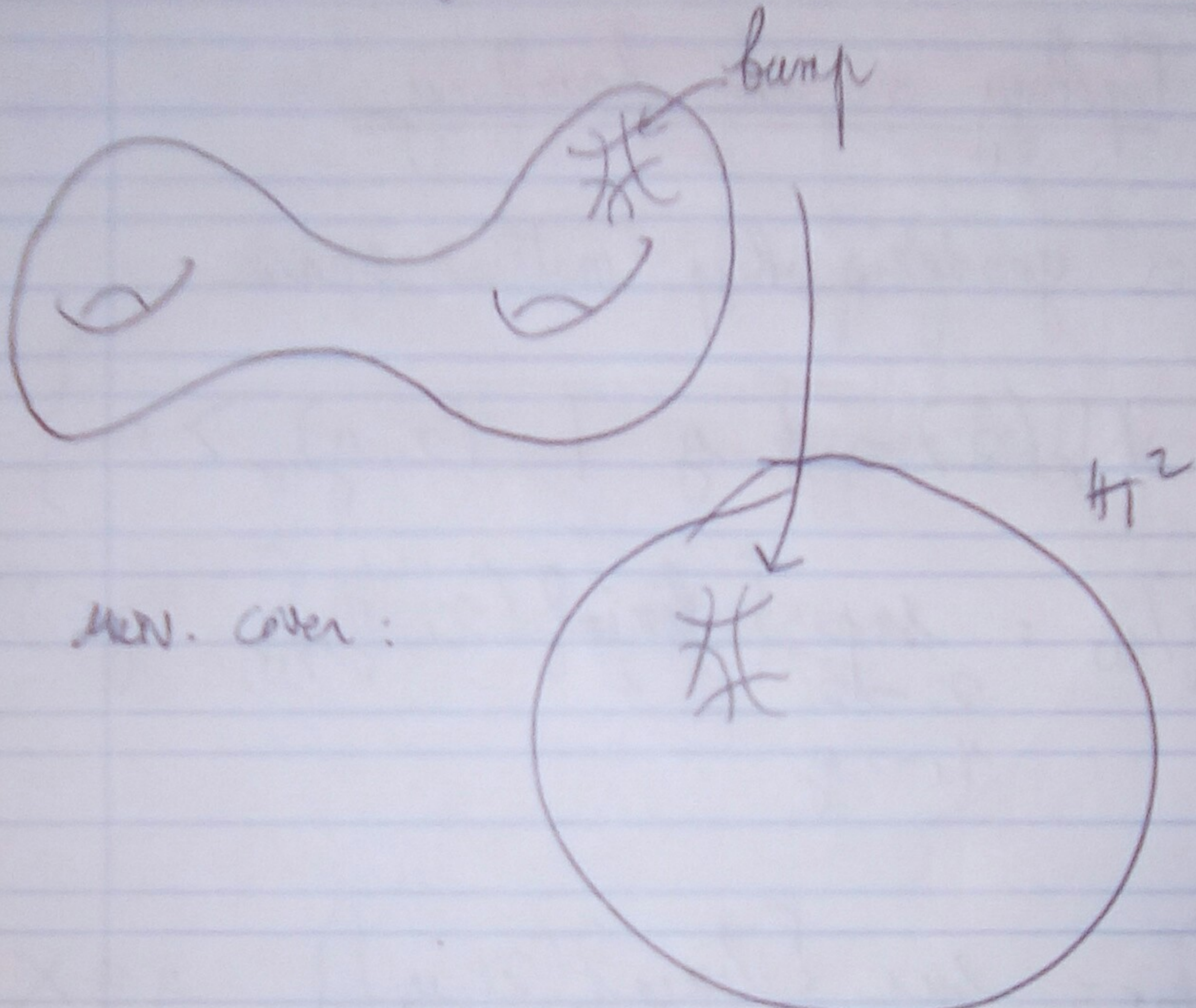
$N_{500}(a^\infty) \ni \{ a^{500}, a^{500}b, a^{500}b^2, \dots \}$



Topology of ∂F_2 is a Cantor set (exercise)

↑ the whole branch starting at a^{500}.

The boundary is a QI-invariant



still qi, so we want that it has the same boundary as $\mathbb{H}^2$.

Def: $f: X \to X'$ qi btw hyperbolic spaces.

use the definition $\partial X = {}^{\text{quasi-geo}}/_{\text{bdd Hausdorff distance}}$

$$\partial f : \partial X \longrightarrow \partial X'$$
$$[\gamma] \longmapsto [f(\gamma)]$$

Prop: Doesn't depend on the choice of γ.

Pf: $\gamma_1 \sim \gamma_2 \Rightarrow d_H(\gamma_1, \gamma_2) < c$

so $d_H(f(\gamma_1), f(\gamma_2)) \leq ac + b$

Let g be a quasi-inverse of f.

$d_H(g \circ f(\gamma), \gamma)$ is bounded,

so $[g \circ f(\gamma)] \sim [\gamma]$

This means, $\partial g \circ \partial f = Id_{\partial X}$

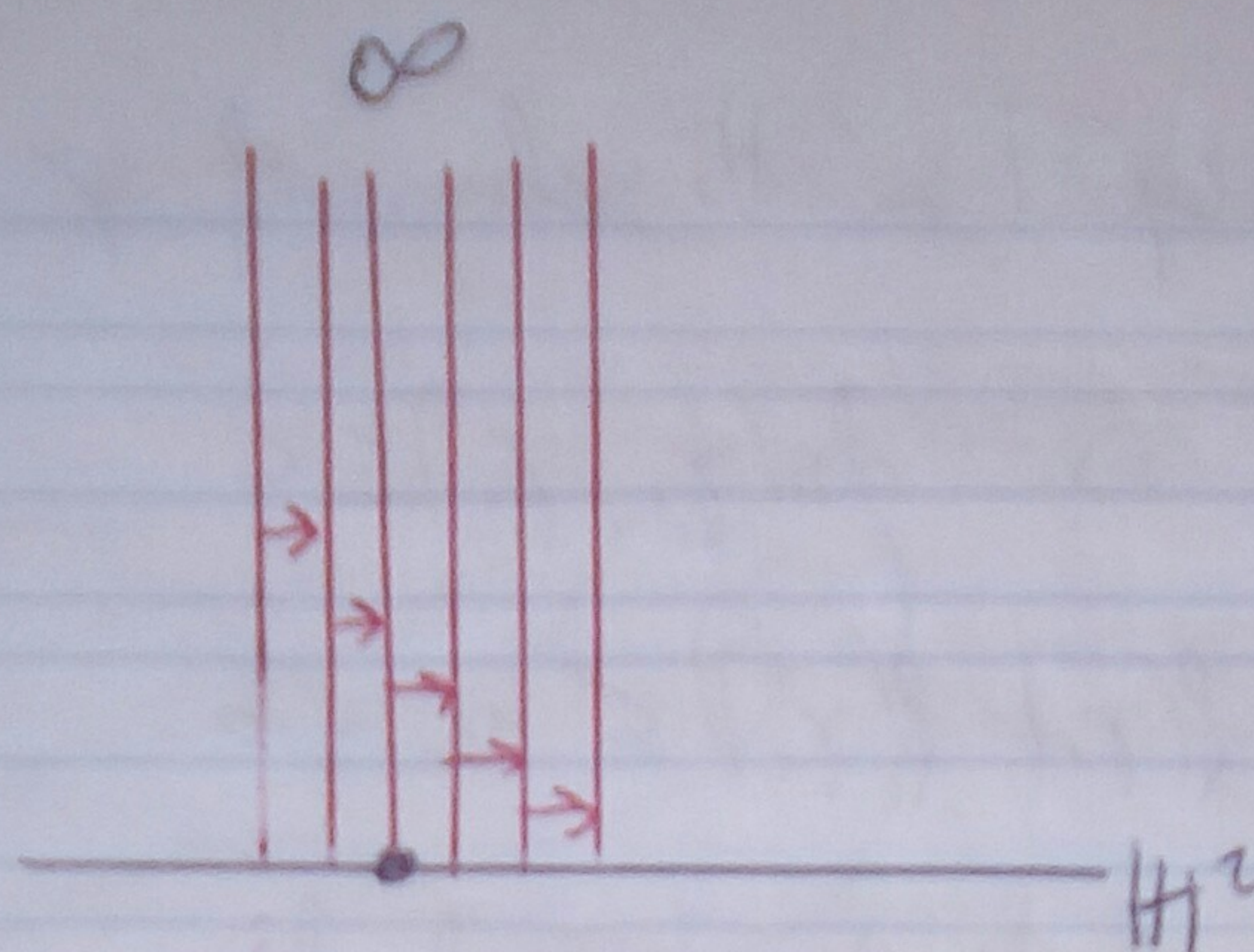
$\partial f \circ \partial g = Id_{\partial X'}$

In particular, ∂f is bijective

Note: a group acting by isometries on X also acts on ∂X by homeomorphisms.

~~xx~~

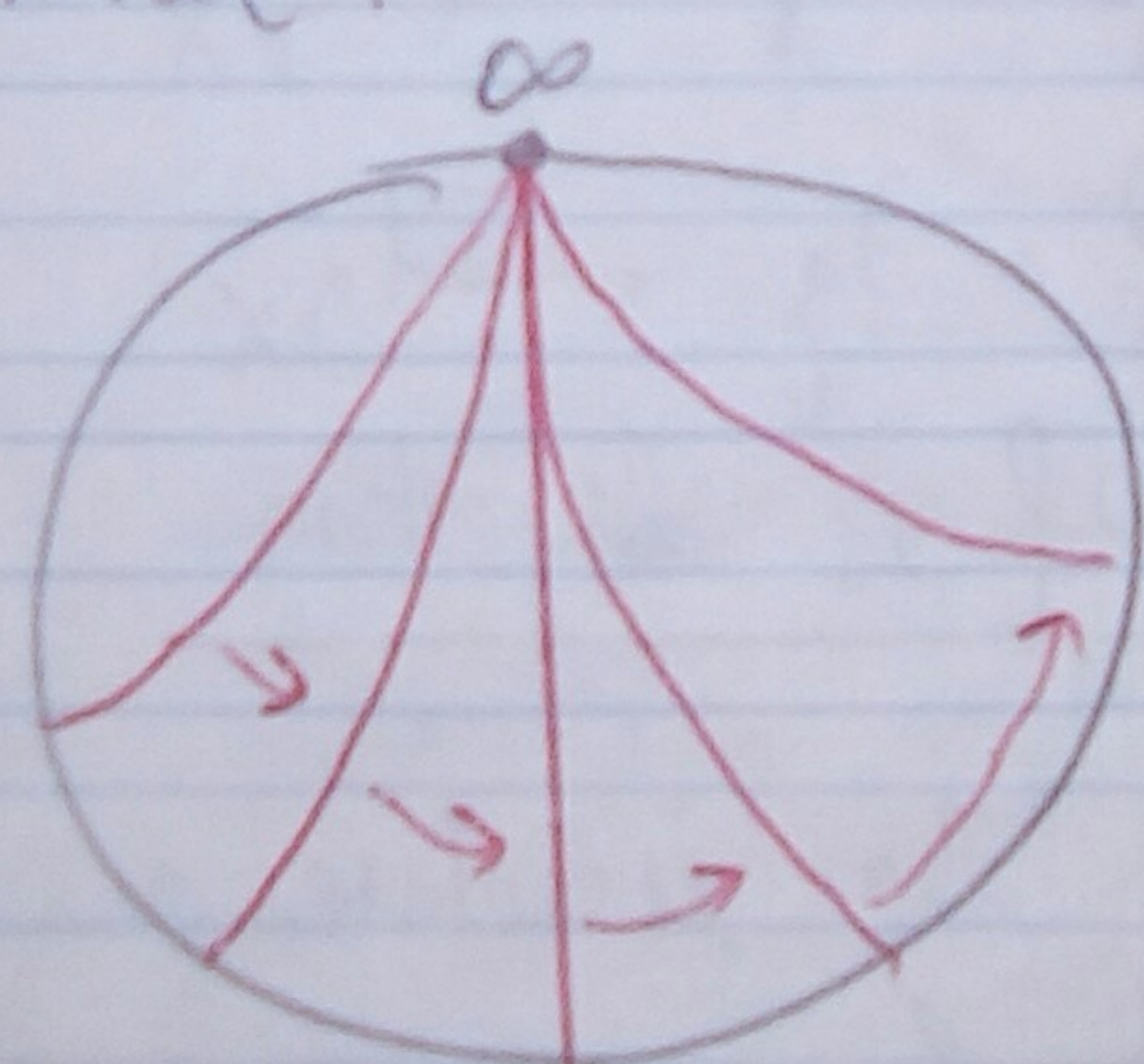
Ex:



$z \mapsto z+1$

$\partial \mathbb{H}^2 \setminus \{\infty\} / G =$ 

circle model:

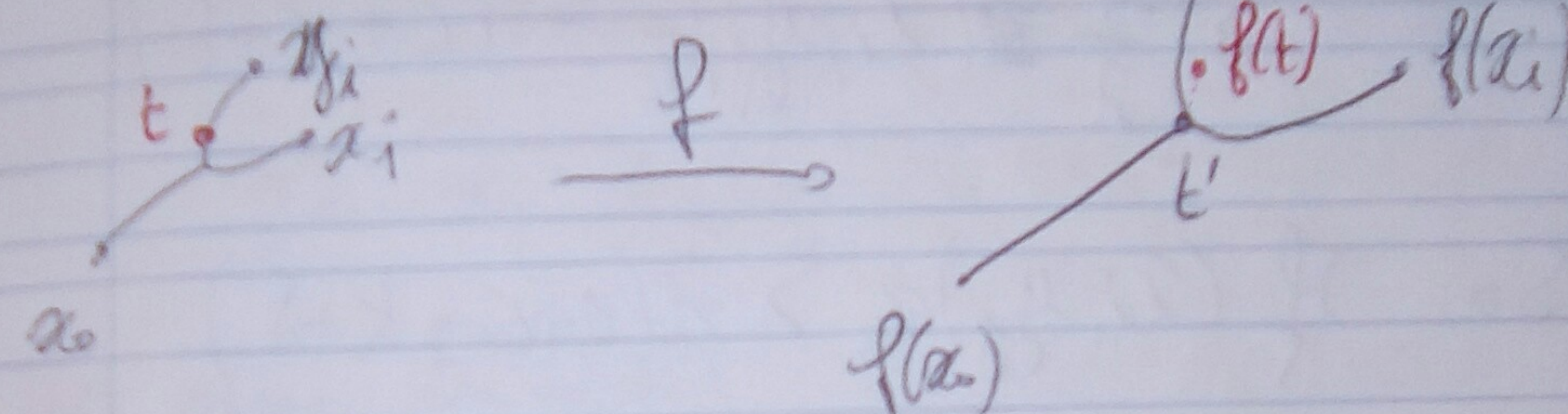


Continuity

$f : X \to X'$ a λ-c quasi-isometry

$x_i \to \bar{x}$

$y_i \to \bar{y}$



in X': $(f(x_i), f(y_j))_{f(x_0)} \geq d(f(x_0), [f(x_i), f(y_j)]) - \delta$

$= d(f(x_0), t') - \delta$

for some $t' \in [f(x_i), f(y_j)]$

$[x_i, y_j]$ maps to a quasi-geodesic

$\exists t \in [x_i, y_j] : f(t)$ close to t'

To check: $(f(x_i), f(y_j))_{f(x_0)} \geq d(f(x_0), f(t)) - K - \delta$

for some $K \geq 0$.

(next page)

$$\geq d(f(x_0), f(t)) - K - \delta$$
$$\geq \frac{1}{\lambda} d(x_0, t) - C - K - \delta$$
$$\geq \frac{1}{\lambda} d(x_0, [x_i, y_j]) - C - K - \delta$$
$$\geq \frac{1}{\lambda} (x_i, y_j)_{x_0} - (C + K + \delta)$$

So: if $(x_i, y_j)_{x_0} > \lambda(n + C + K + \delta)$

then, $$(f(x_i), f(y_j))_{f(x_0)} > \frac{1}{\lambda}\big(\lambda(n + C + K + \delta)\big) - C + K + \delta$$
$$= n$$

This shows continuity.

Since ∂f is bijective (and X, X' are compact Hausdorff),

<u>∂f is a homeomorphism.</u>

$\boxed{\text{Why this is great :}}$

If G is hyperbolic, and acts geometrically on X and on Y.

Then, $\partial X \underset{\text{homeo}}{\simeq} \partial Y$.

We can define $\partial G \simeq \partial X \simeq \partial Y$.

This opens the field : look at ∂X, what possible G have this boundary ?

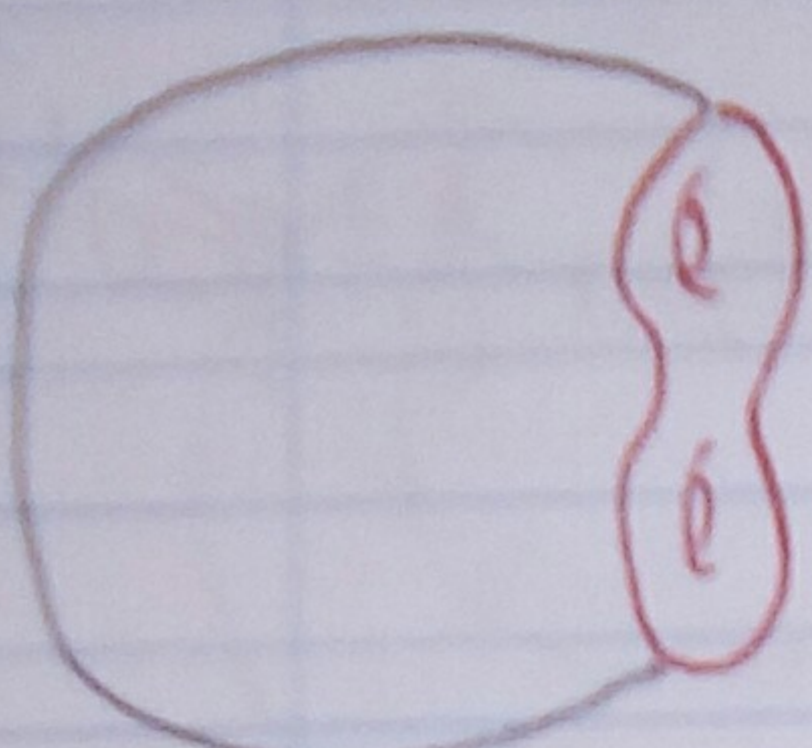
- $\partial G = \cdot \ \cdot \iff G$ is virt. $\mathbb{Z}$
- ∂G is a Cantor set $\iff G$ is virt free (of rank ≥ 2)
- $\partial(\pi_1(S_g)) = S^1$

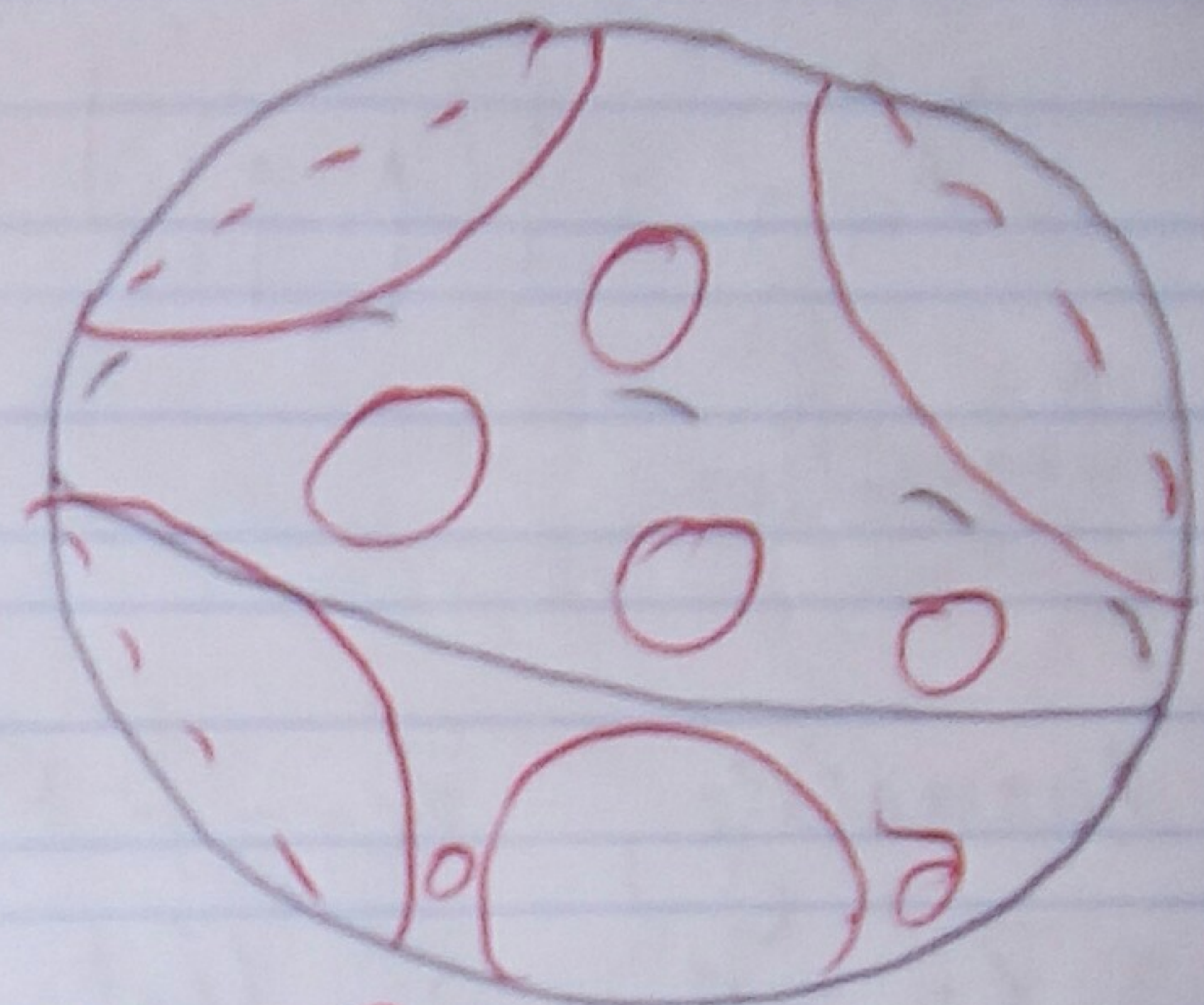
Theorem (Tuka, Gabai, ...)

if $\partial \Gamma \simeq S^1$, then Γ is virt. Fuchsian

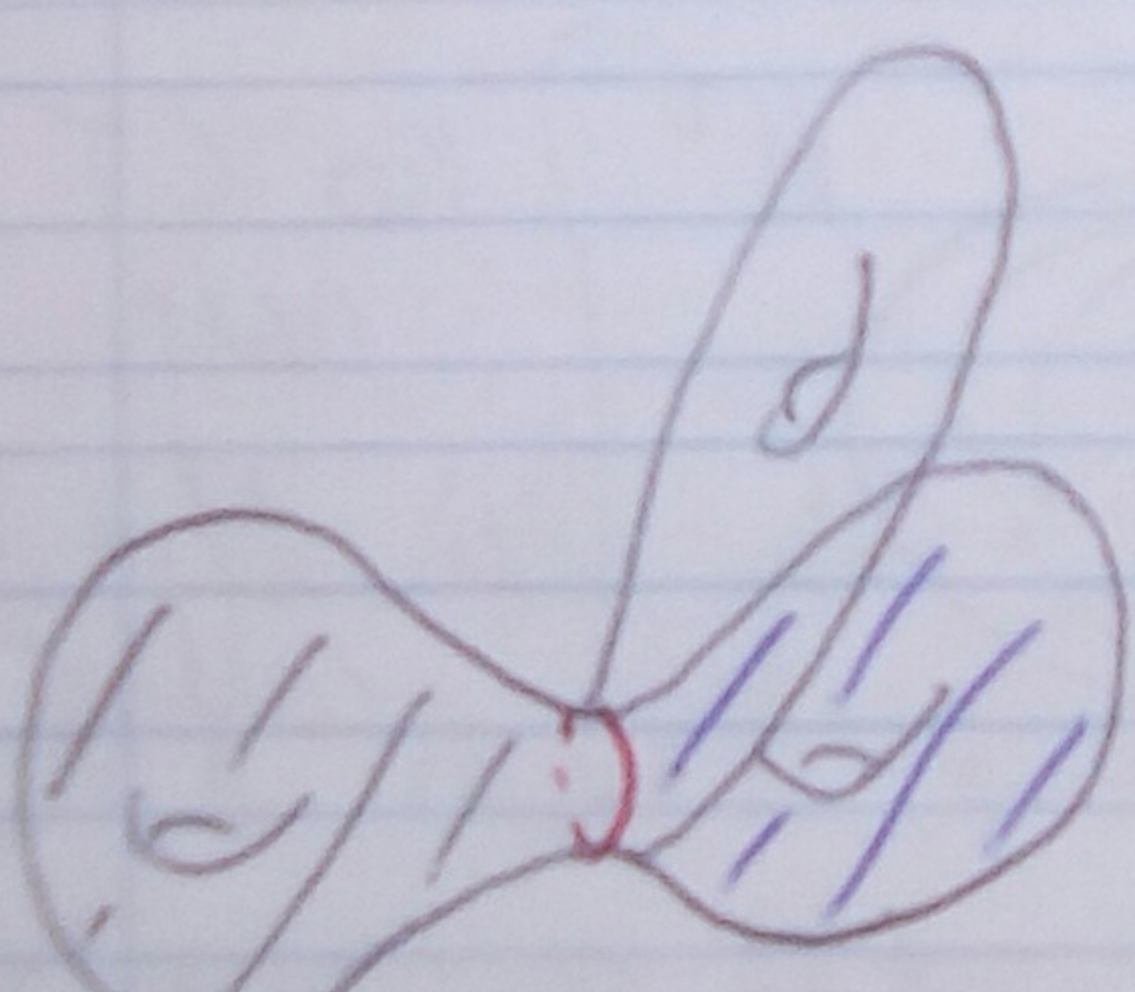
What other boundaries can occur?

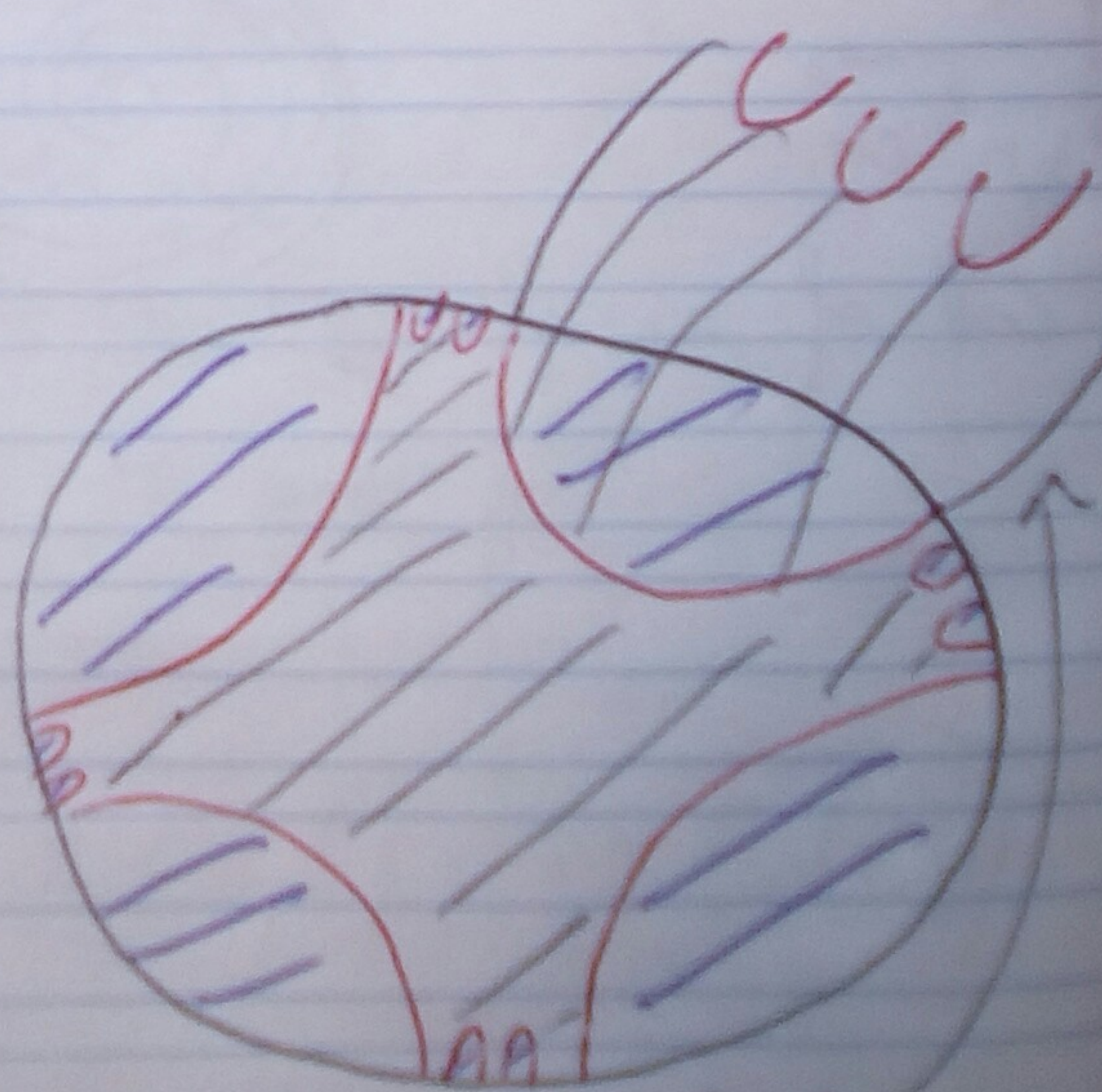
- ∂($\mathbb{H}^3$ w/ S_g boundary) = Sierpinski carpet





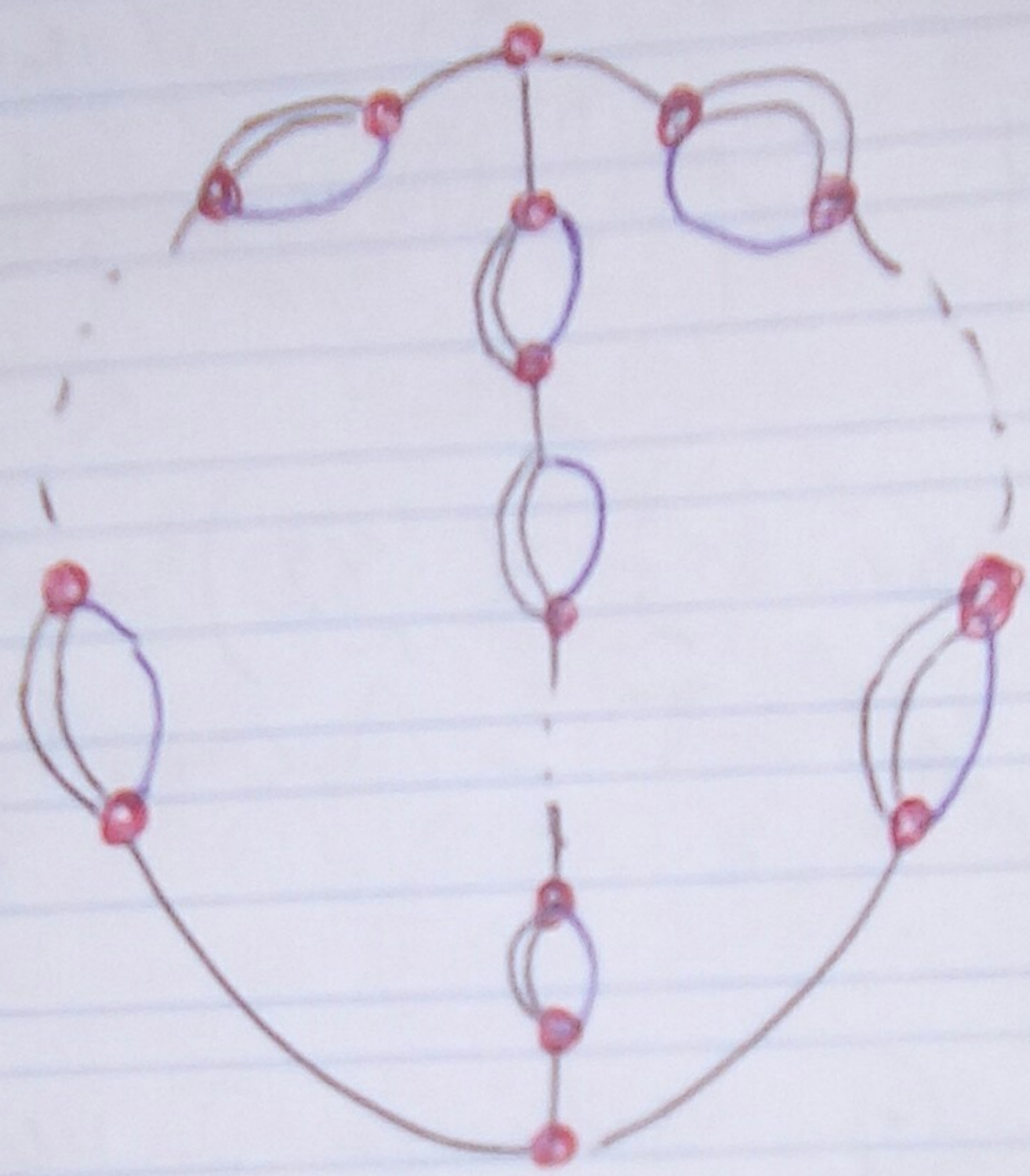
∞ many circles
(lifts of)

- 



glue one "sheet"
on each lift of the red curve

The boundary looks like this:



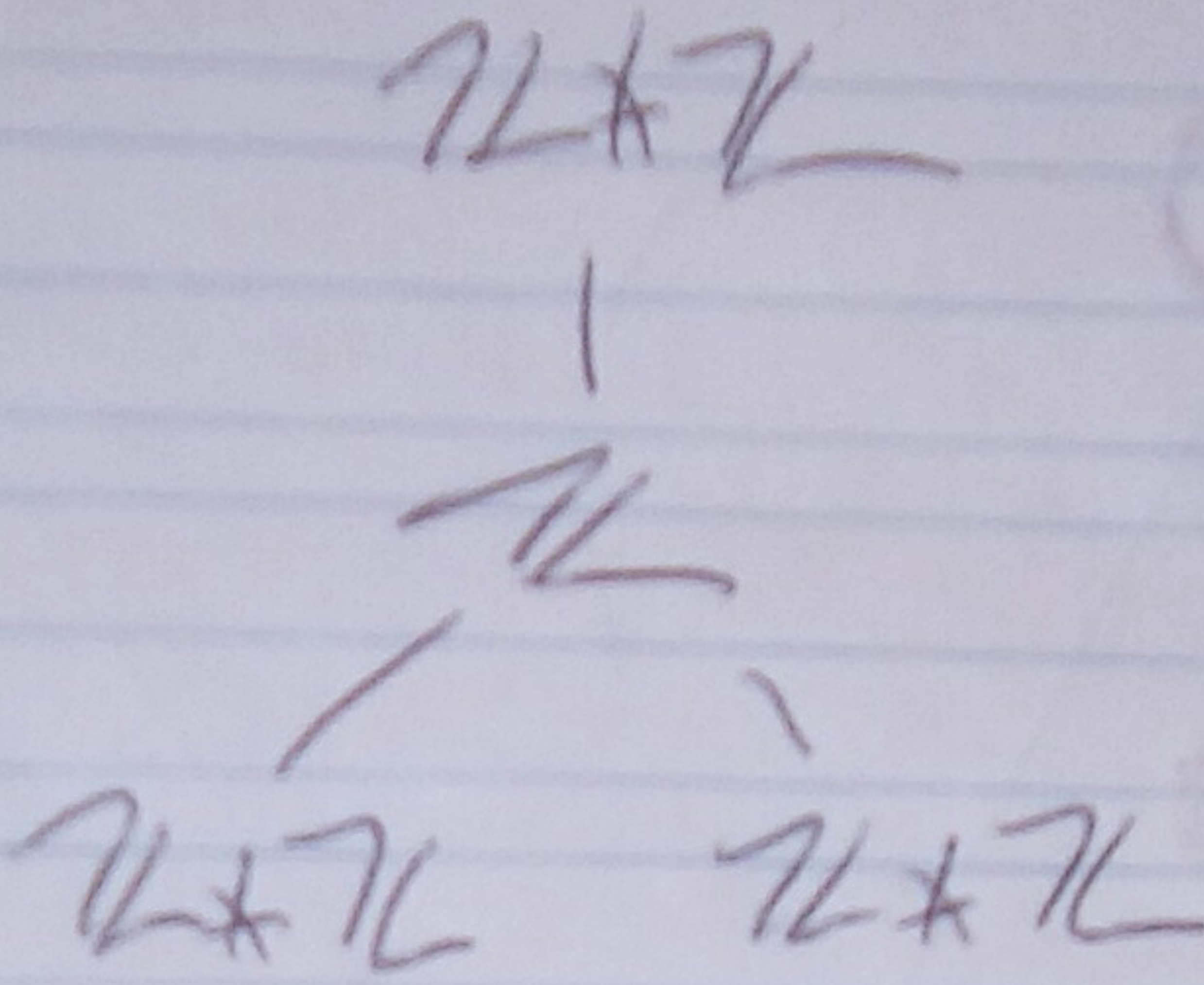
Fact: . it is connected
. it doesn't have any cutpoint
↳ (as every boundary of a hyp. gp.)

But it has local cutpoints (red vertices)

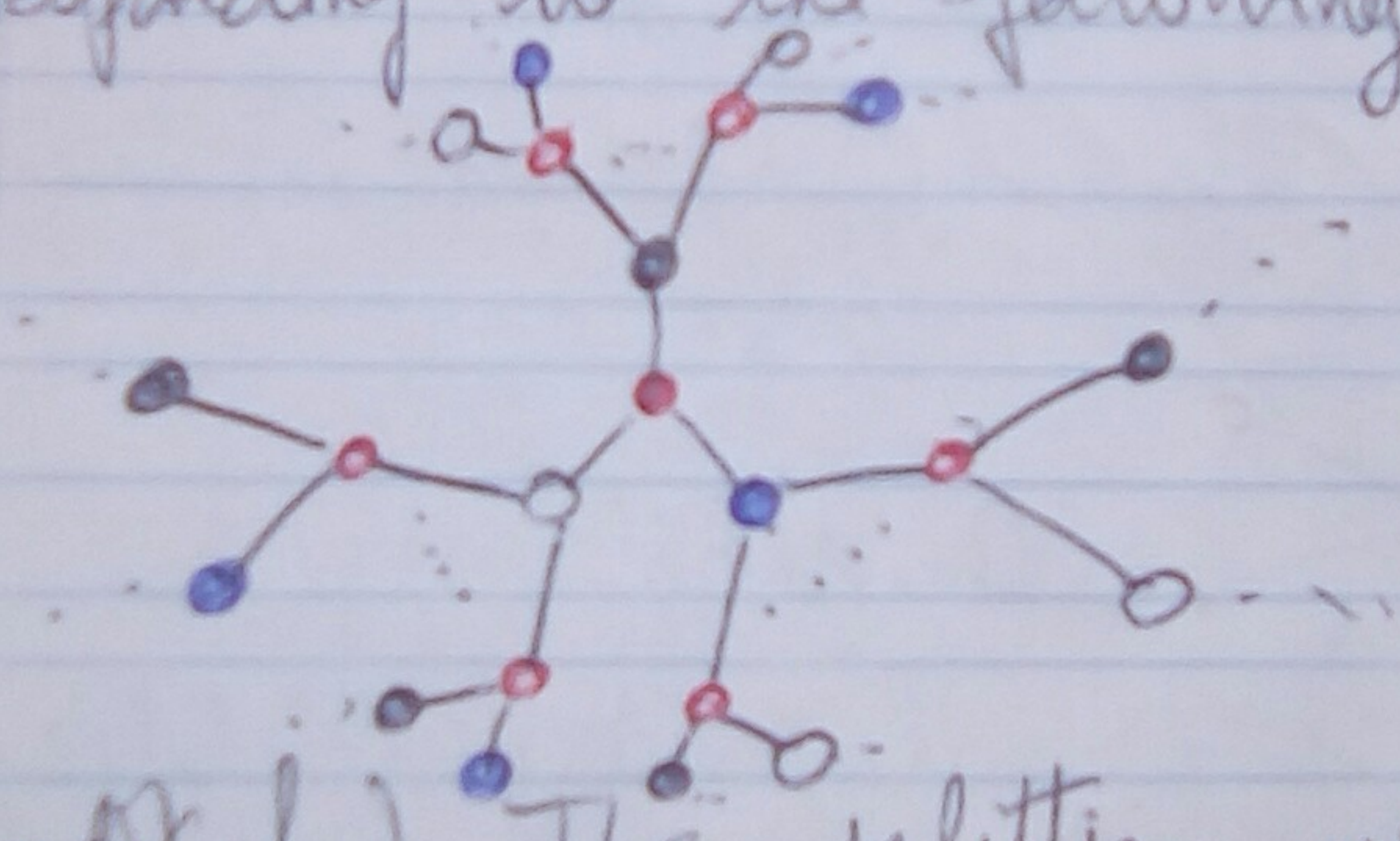
From Bowditch, ∃ local cutpoint ⟺ splits over $\mathbb{Z}$

Pairs of cutpoints give information on splitting over $\mathbb{Z}$.

graph of groups



Corresponding to the following Bass-Serre tree:



(Bowditch) This splitting yields vertex stabilizers:

- elementary ($\mathbb{Z}$)
- hanging Fuchsian
- rigid (don't admit any more splitting over $\mathbb{Z}$)