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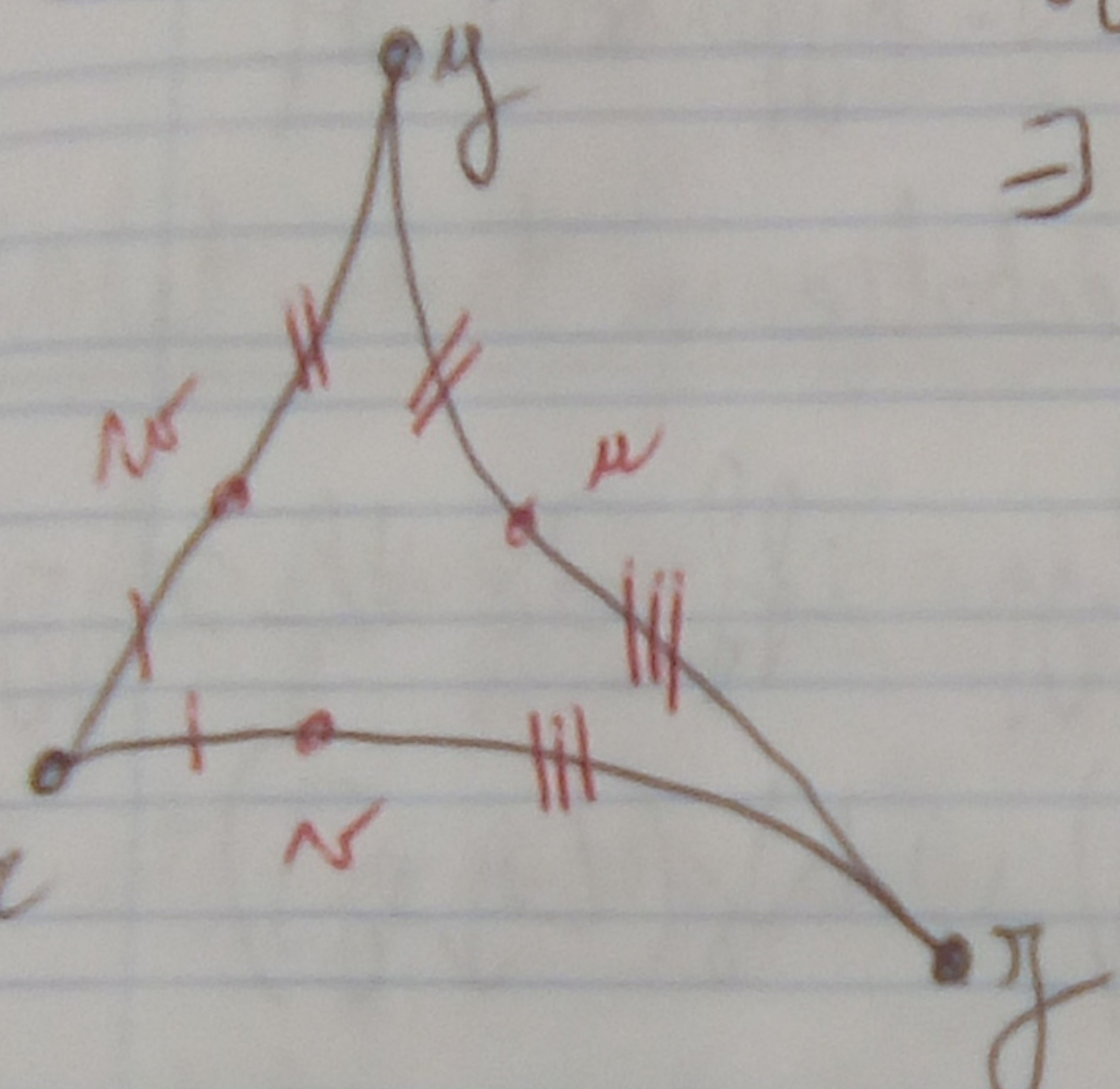
$\boxed{\text{"from 0 to infinity"}}$

X : proper geodesic metric space

<u>Lemma/Definition</u>: xyz a (geodesic) triangle in X

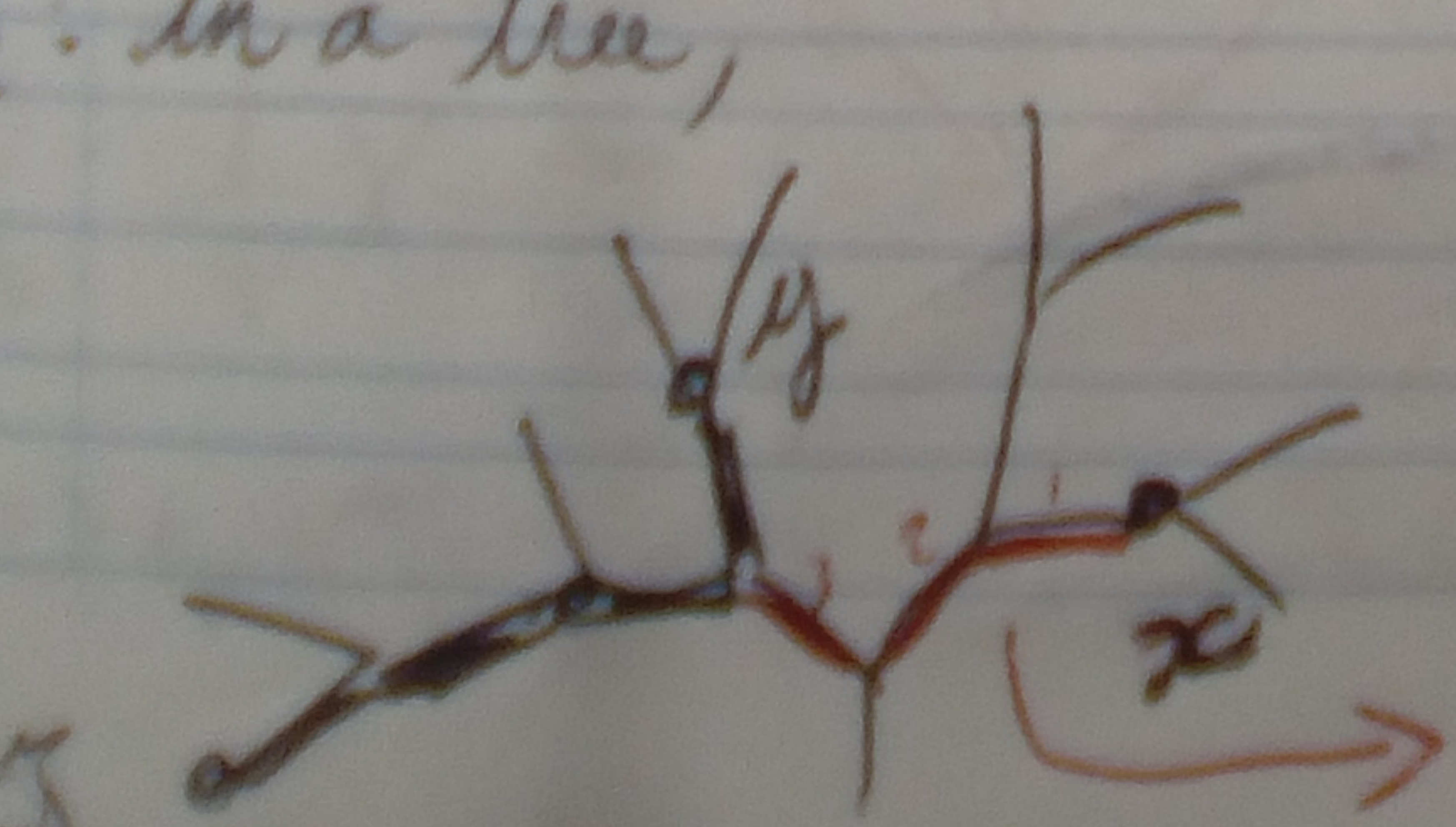


$\exists\, w \in [x,y]$, $u \in [y,z]$, $v \in [x,z]$ such that

$$d(x,w) = d(x,v)$$
$$d(y,w) = d(y,u)$$
$$d(z,v) = d(z,u)$$

$$(y \mid z)_x = (y, z)_x = \frac{1}{2}\big(d(x,y) + d(x,z) - d(y,z)\big)$$

<u>Ex</u>: in a tree,



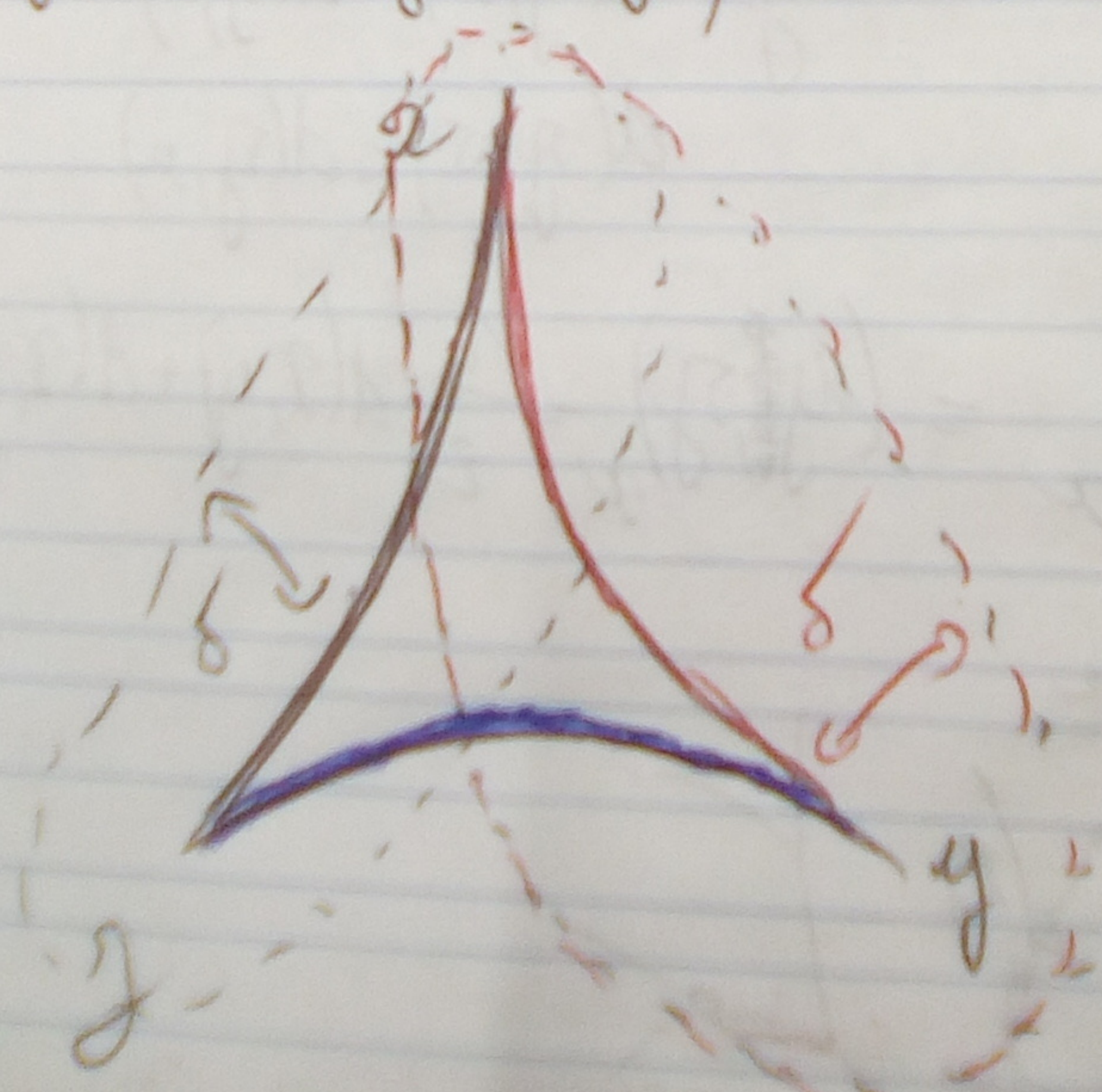
$\rightarrow (y \mid z)_x = 3$

Definition: X is **δ-hyp** if for any Δ xyz and $y' \in [x,y]$, $z' \in [x,z)$ with $d(x,y') = d(x,z') \leq (y,z)_x$ then $d(y',z') < \delta$

Definition: X is Gromov-hyperbolic if $\exists \delta$: X is δ-hyperbolic

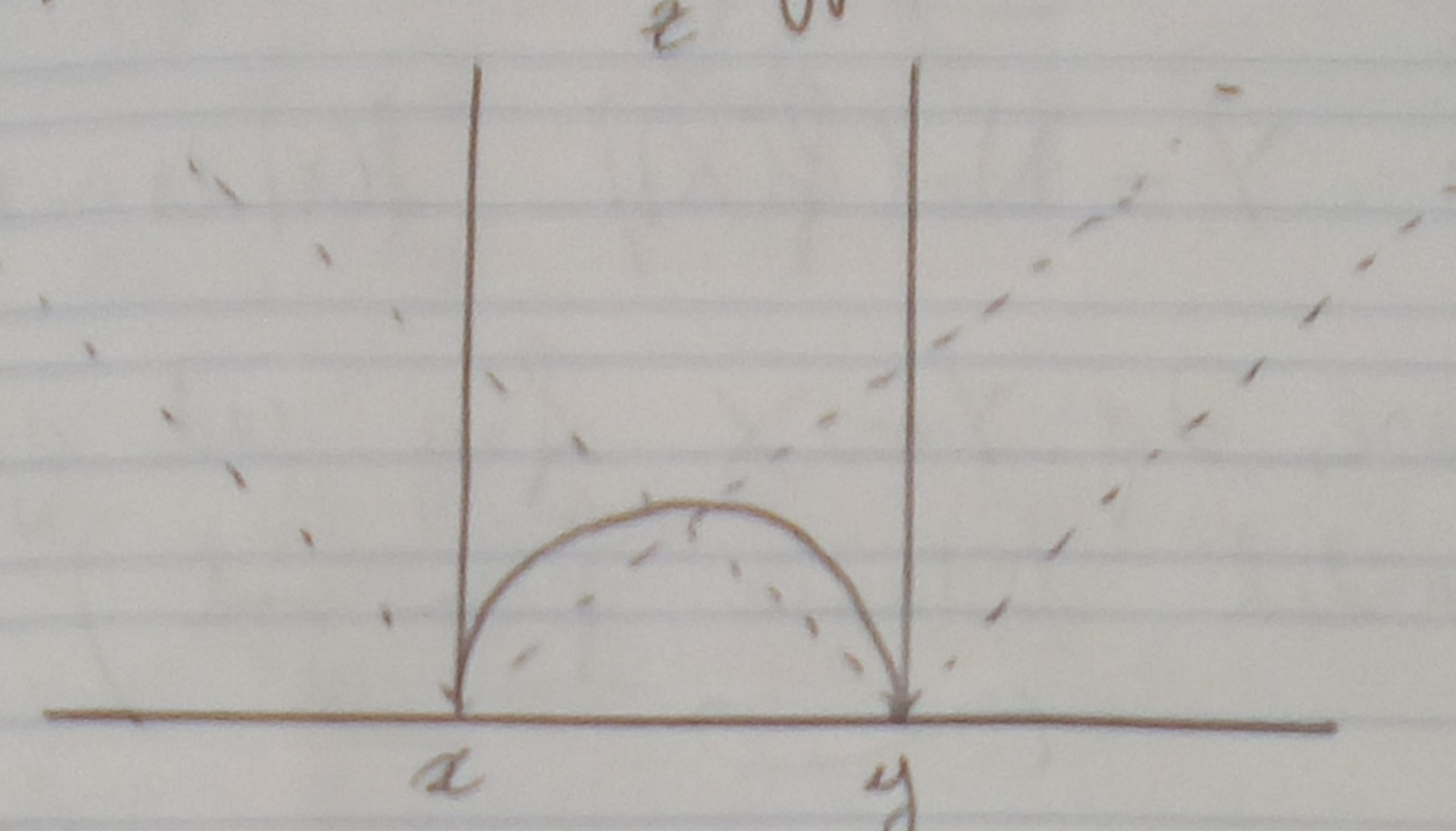
Ex: X is Gromov-hyp iff $\exists \delta$ $\forall \Delta$ xyz

$$[x,y] \subseteq N_\delta([x,z]) \cup N_\delta([y,z])$$



__Ex__: $\mathbb{H}^2$ is Gromov-hyperbolic.

(ideal triangle)



What about groups?

A group is (word, Gromov) hyperbolic if it acts geometrically on a (proper, geodesic) hyperbolic metric space

↳ cocompactly + by isometries

Def: $f: X \to Y$ quasi-isometric embedding

$\exists a \geq 1, b \geq 0 \quad \frac{1}{a} d(x,x') - b \leq d(f(x), f(x')) \leq a\, d(x,x') + b$

(end of definition:)

f is a quasi-isometry if, additionally,

$\exists c \geq 0 \quad Y = N_c(f(X))$ "$f(X)$ is a net"

(in that case, $\exists g : Y \to X$ $f \circ g$ and $g \circ f$ are at bounded distance from Id)

$\sim$

Stability of quasi-geodesics (Morse lemma)

X δ-hyp. space

Thm: $c : [a,b] \to X$ a (λ, ϵ) quasi-geodesic from p to q.

$[p,q]$ a geodesic from p to q

Then: $d_{Haus}([p,q], im(c)) \leq R = R(\delta, \lambda, \epsilon)$

"geodesics are close".

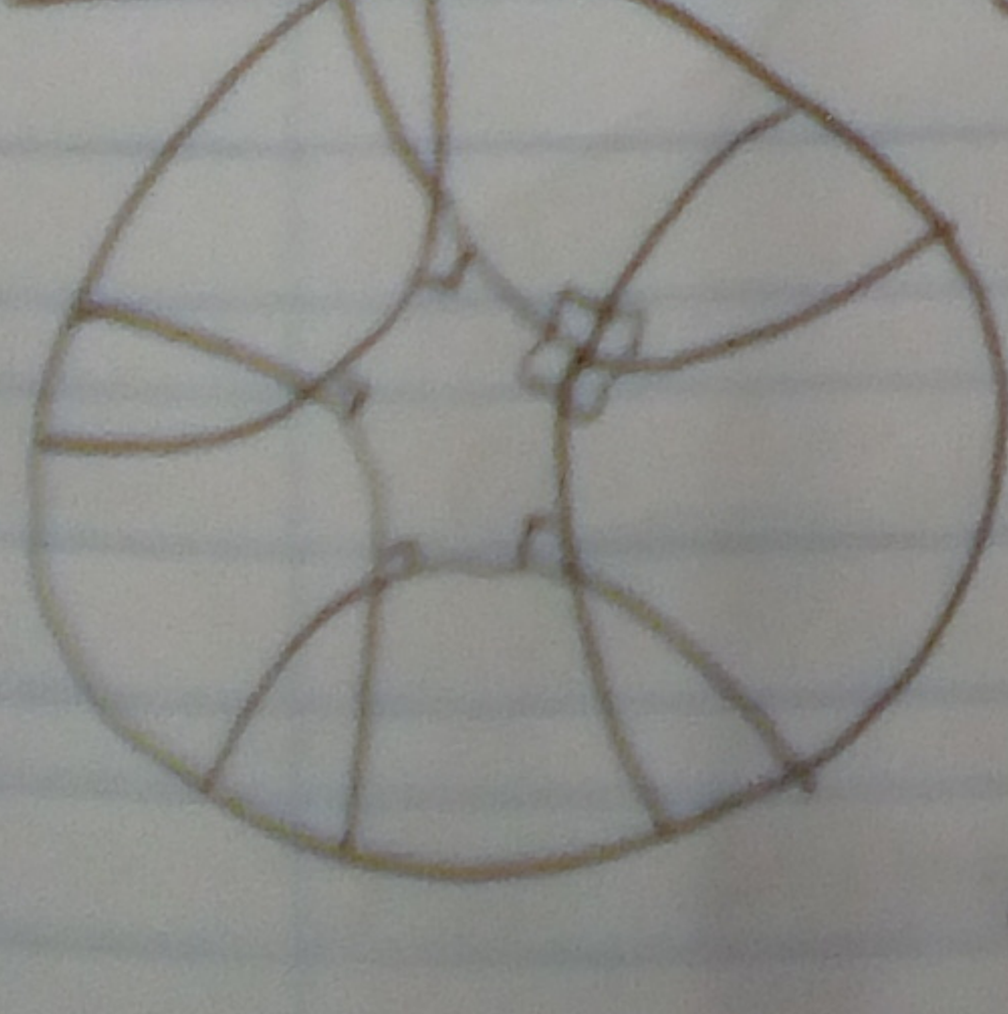
$\sim$

Exercise 1) use this to show that if X is quasi-isometric to Y, and X is Gromov hyperbolic, then so is Y.

2) if G acts geometrically on some hyperbolic space, then any space it acts on geometrically is also hyperbolic.

$\rightsquigarrow$ G is hyperbolic iff Cayley(G, S) is hyperbolic for some finite generating set S of G.

Ex: right-angled pentagon in $\mathbb{H}^2$.



G = group of reflections in the sides

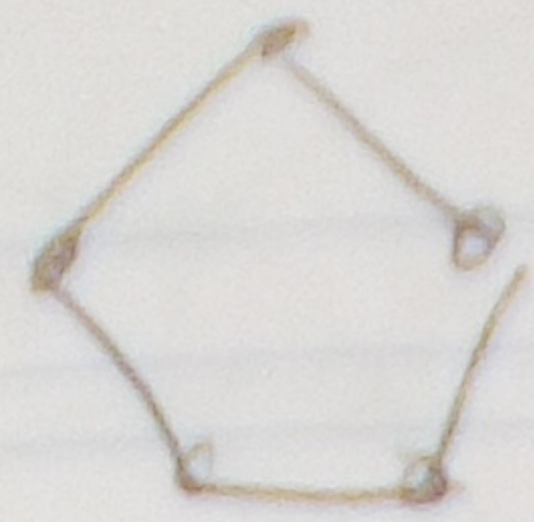
The image will tile $\mathbb{H}^2$

The group acts by isometries on $\mathbb{H}^2$

$\longrightarrow$ This is a hyperbolic group

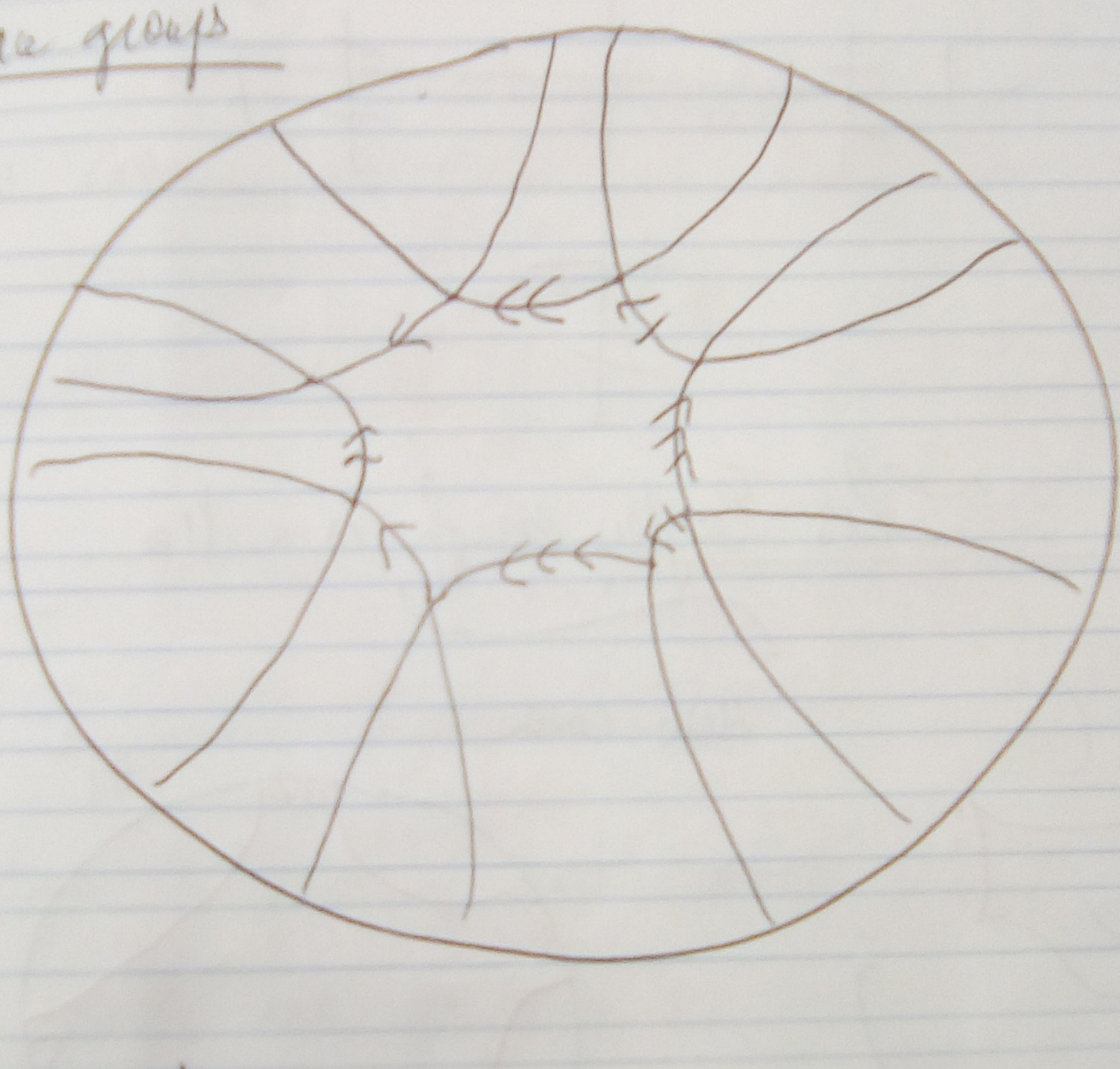
RACG $\langle v_i \in V \mid v_i^2 = 1 \quad v_i v_j = v_j v_i \text{ iff } \{i,j\} \in E \rangle$

(graph V, E next page)



: the graph associated with the RACG.

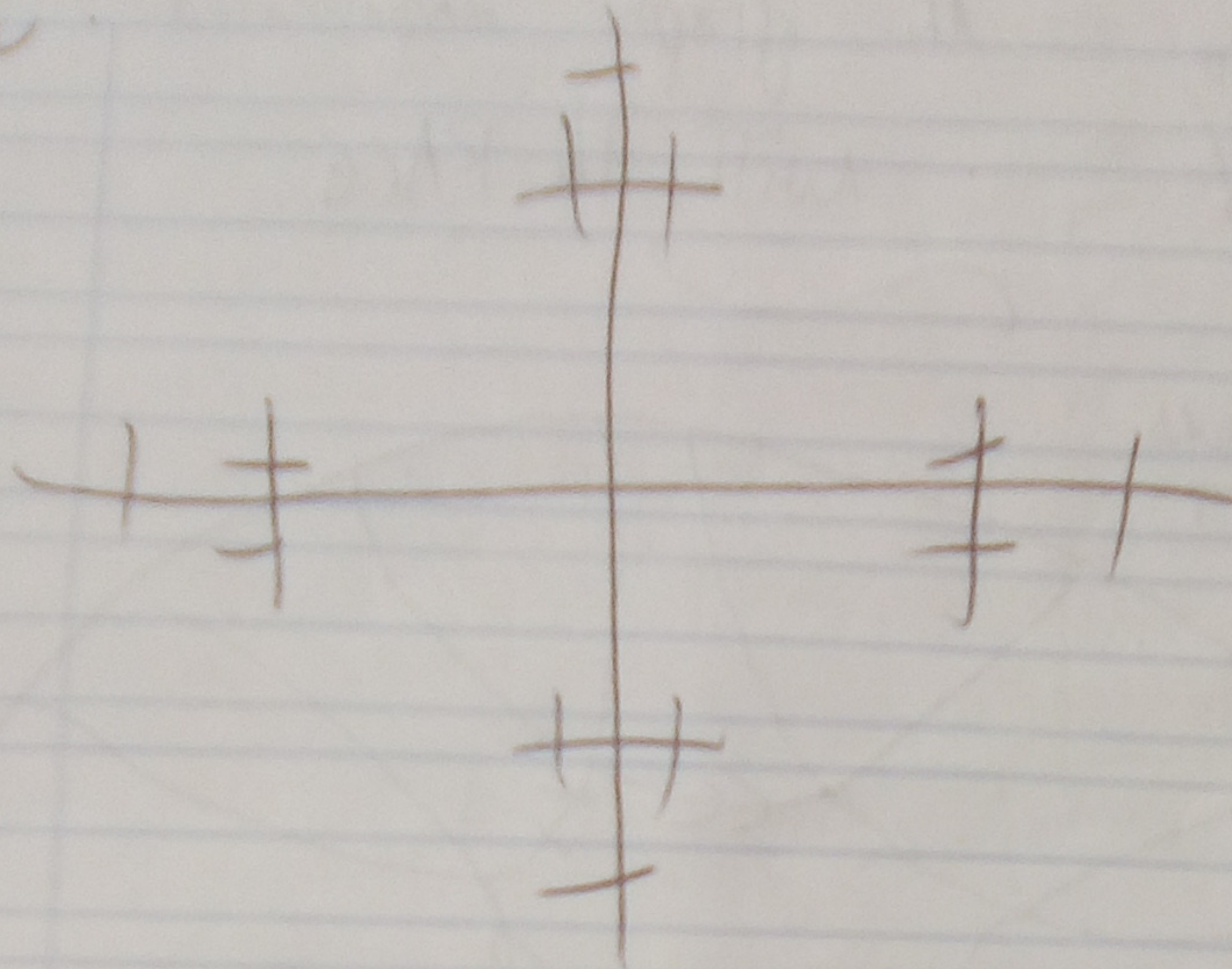
Surface groups



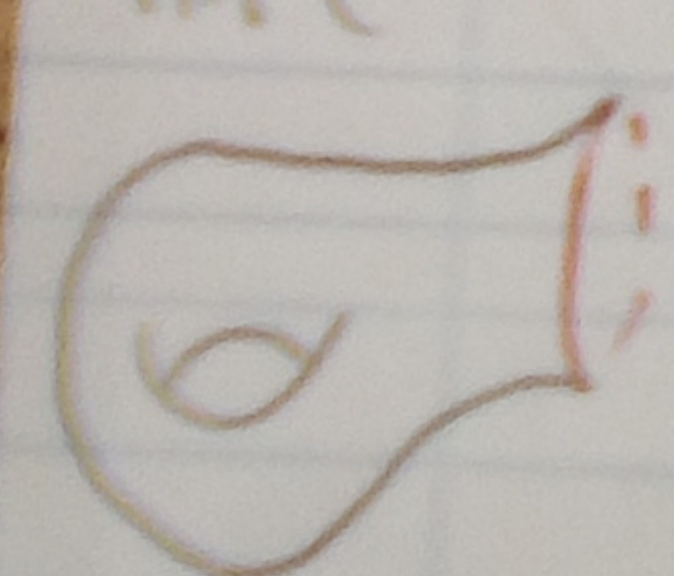
∃ a hyperbolic octagon w/ angles $2\pi/8$

→ surface groups are (almost always) hyperbolic

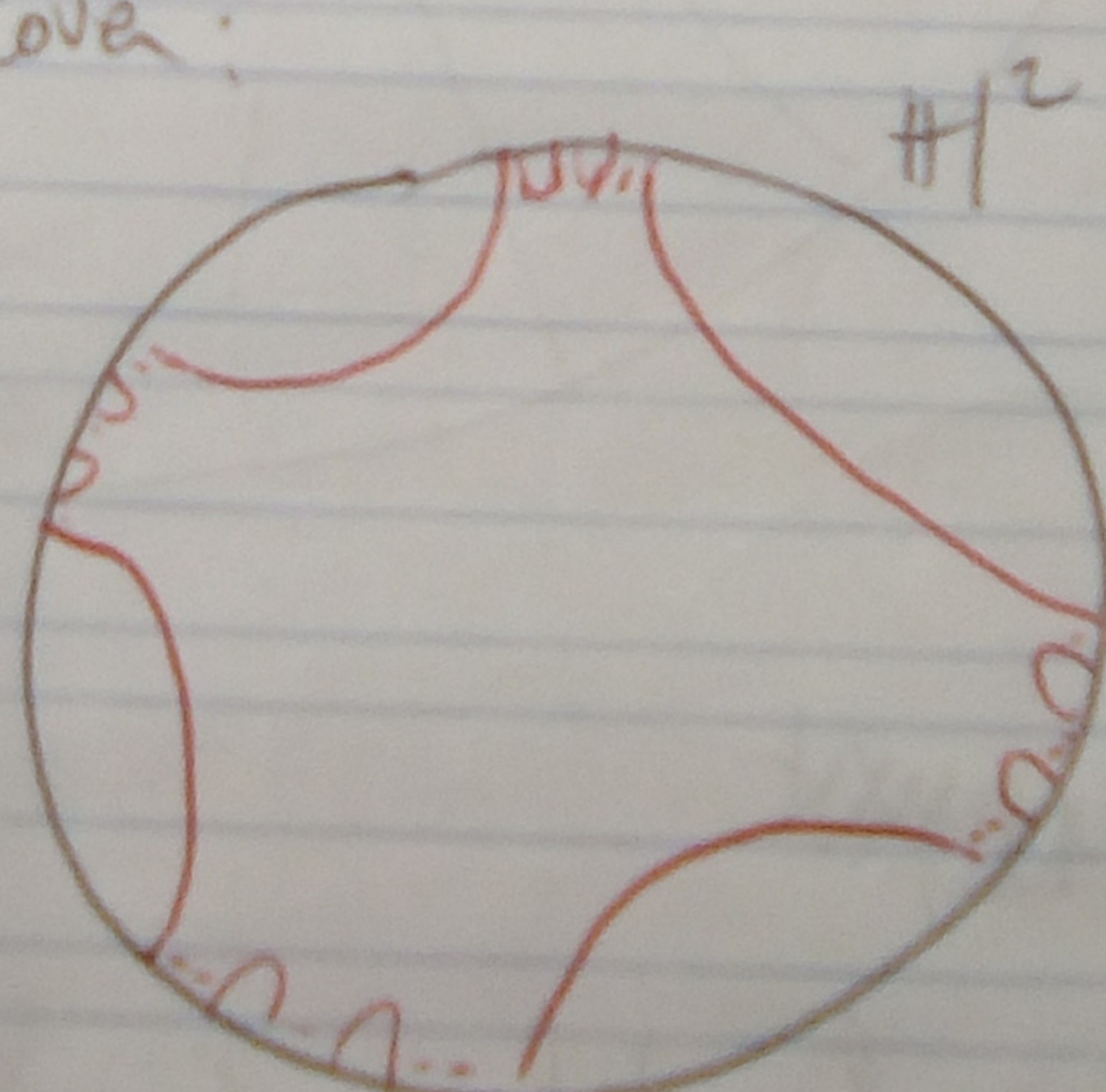
F_2:



This Cayley graph is a tree.

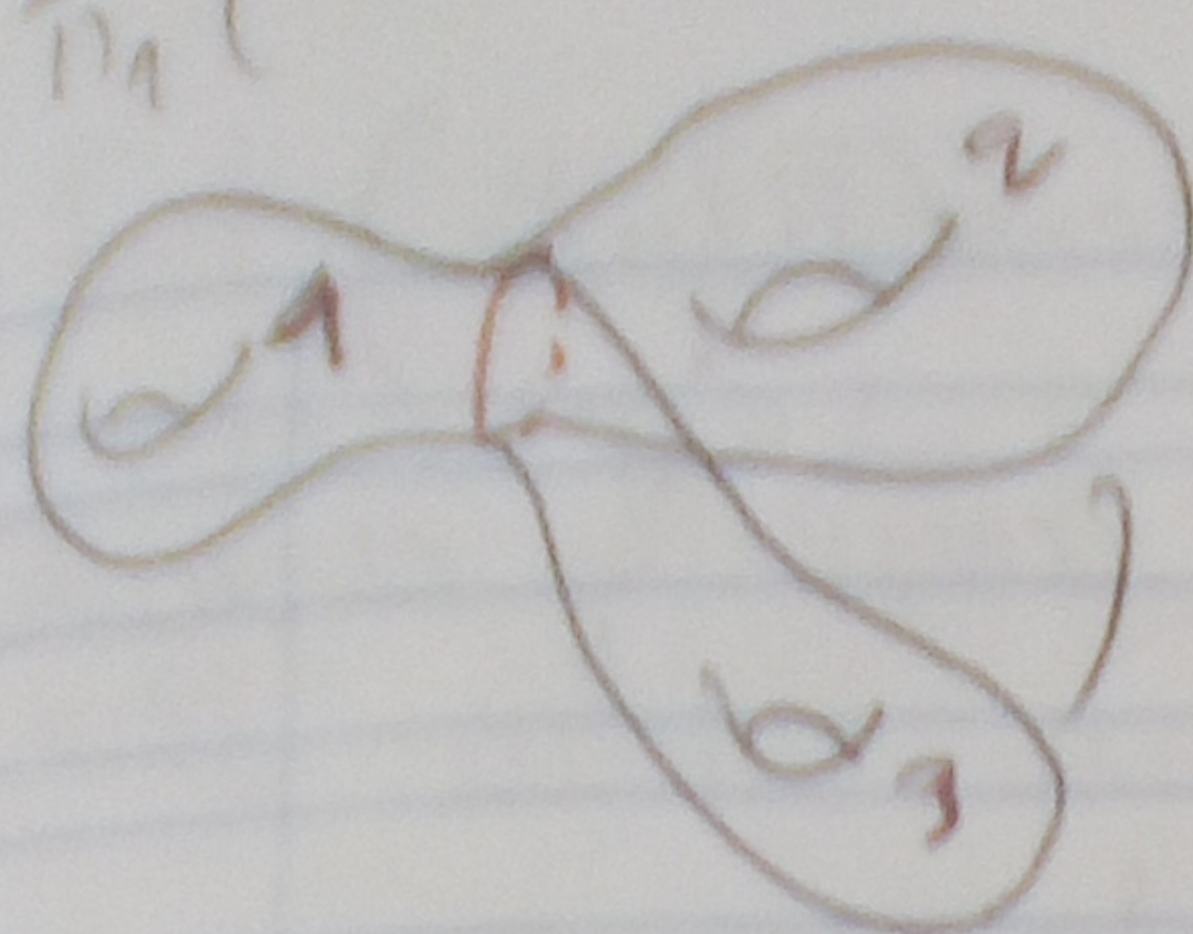
$\pi_1($  $)$

Univ. cover:

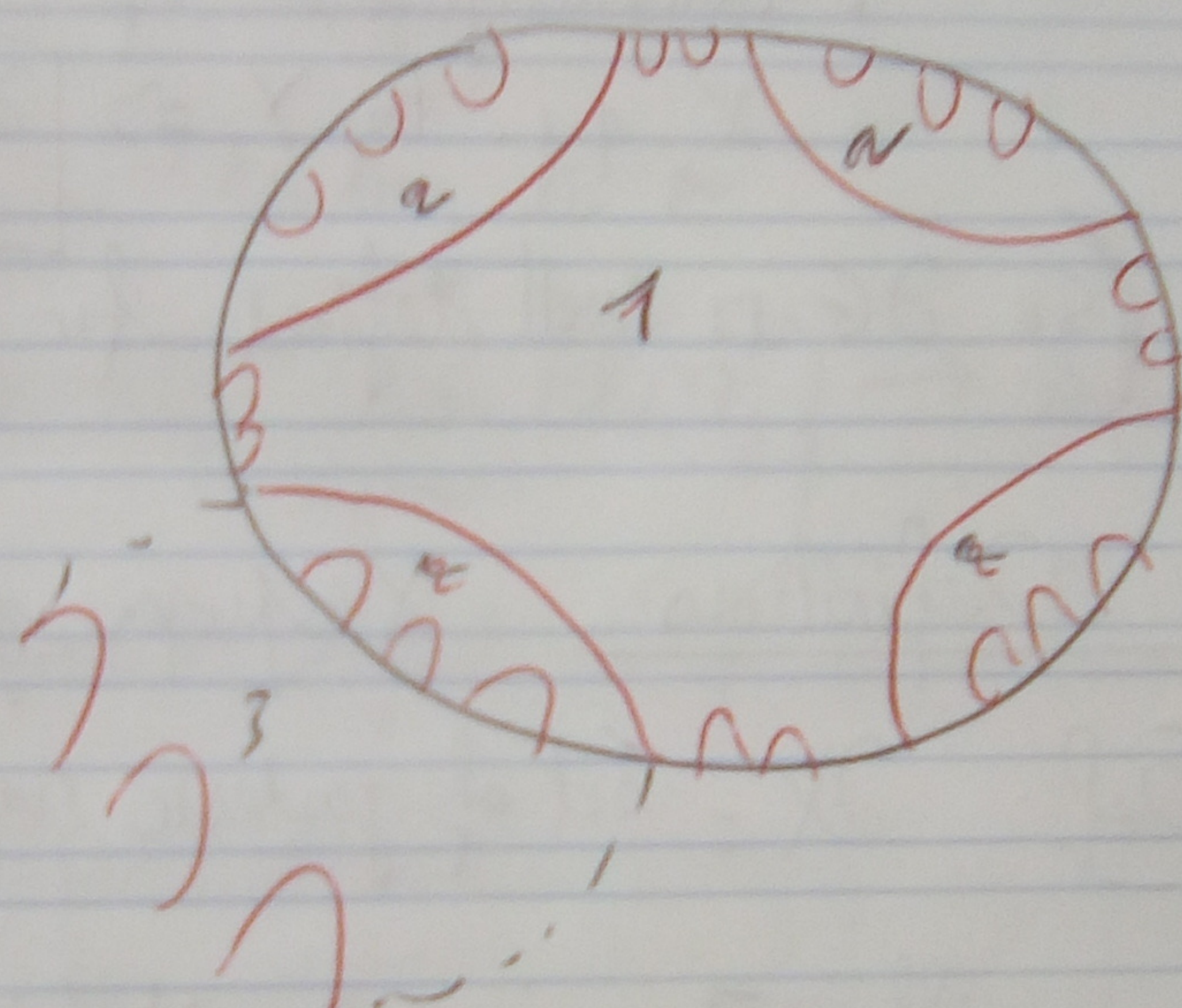


Convex subset of $\mathbb{H}^2$, it is δ-hyperbolic.

$\pi_1($



univ. cover:



(glue a piece of univ. cover of 3 along each red geodesic).

This is a hyperbolic space; so $\pi_1(\ldots)$ is a hyperbolic group.

Boundary? X hyperbolic (infinite)

∂X topological object canonically associated to X.

- Compactification of X
- X q.i. to $Y \Rightarrow \boxed{\partial X \underset{\text{homeo}}{\simeq} \partial Y}$

(So ∂G is well defined for G hyperbolic group)

3 Definitions

X proper, geodesic, hyperbolic

Def: ∂X = set of geodesic rays from x_0 $/\sim$

$\gamma \sim \gamma'$ iff: $\exists c$ $im(\gamma) \subset N_c(im(\gamma'))$
$im(\gamma') \subset N_c(im(\gamma))$

Alternative definition:

∂X = set of quasi-geodesic rays $/\sim$

NB: $\sim \Leftrightarrow$ bounded Hausdorff distance

Fact: X Gromov hyp. space.

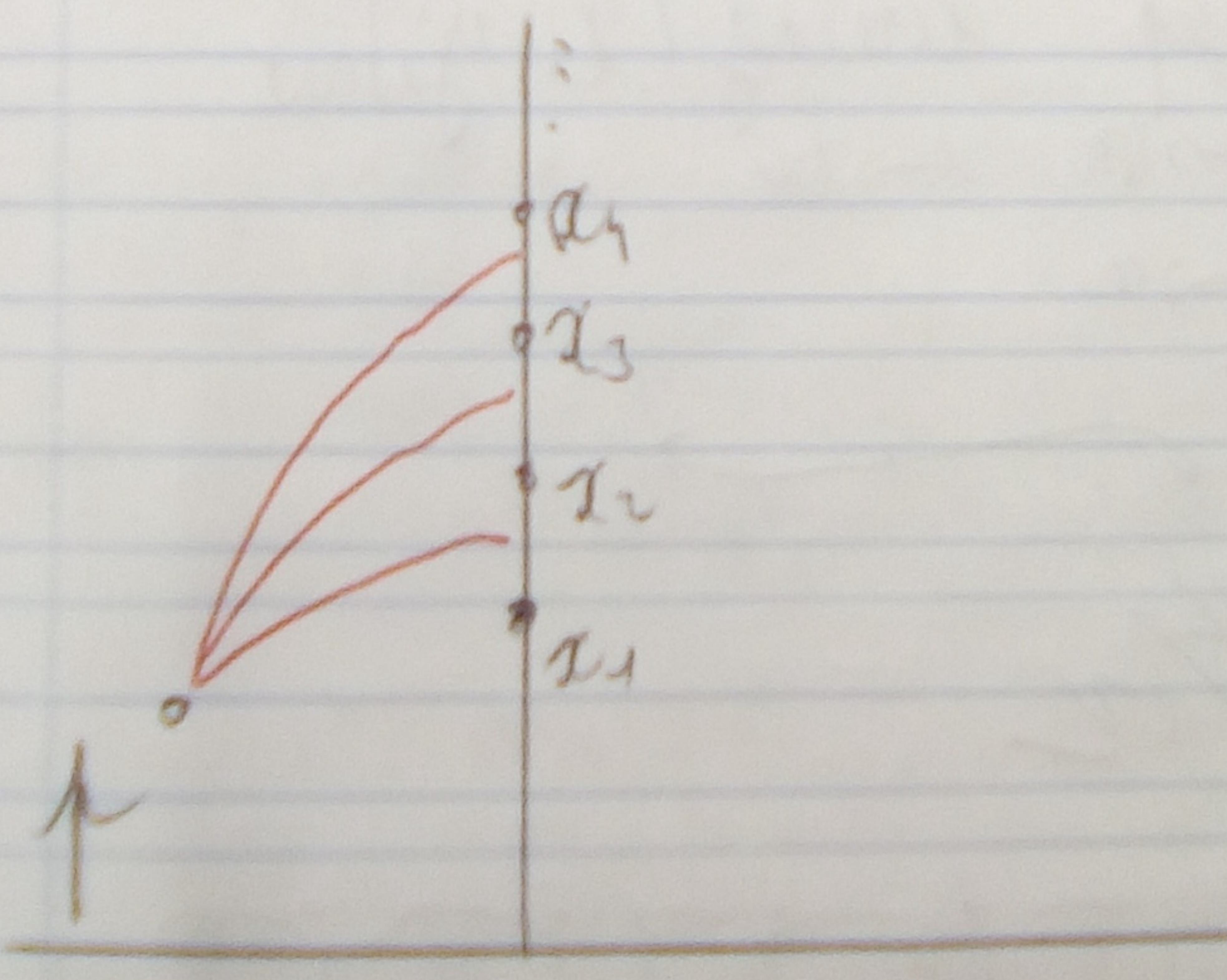
$\exists \delta \quad (x,y)_p \leq d(p,[x,y]) \leq (x,y)_p + \delta$

- The Gromov product approximates the distance to $[x,y]$.

Def: sequences "tending to ∞"

$(x_i) \to \infty$ iff $\lim\limits_{i,j\to\infty} (x_i, x_j)_p \to \infty$

(exercise: doesn't depend on p)



Equivalence: $(a_i) \sim (b_i)$ if $\liminf\limits_{i\to\infty} (a_i, b_i)_p = \infty$

Rough idea why these are the same

geodesic ray @ x_0 $\in$ {quasi-geodesic rays}

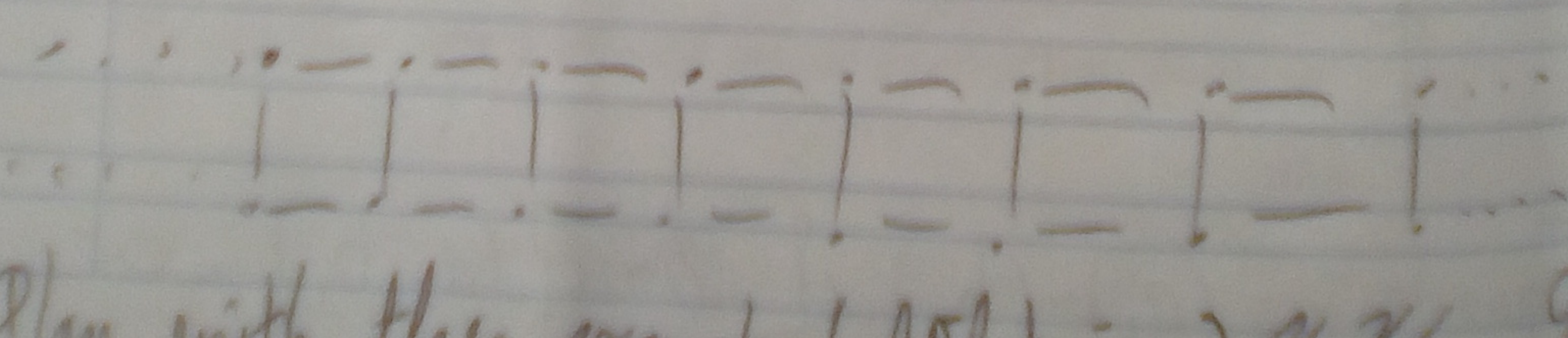
γ quasi-geodesic ray : $(\gamma(i))_{i\in\mathbb{N}}$ sequence tending to ∞.

Topology

$$x_i \longrightarrow m \qquad m = [x_i]$$

$$(m,n)_w = \sup_{\substack{x_i \to m \\ y_i \to n}} \liminf_{i\to\infty} (x_i, y_i)_w$$

Ex : $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$



Play with these examples! What is $\partial \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$?