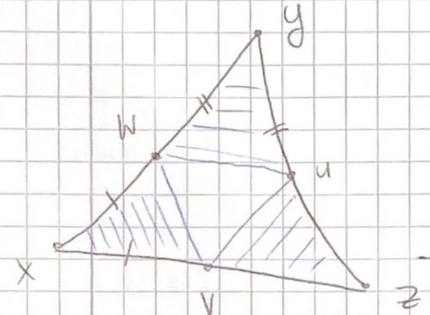
Genevieve Walsh Lecture 1 "From 0 to ∞"

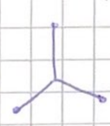
X proper geodesic metric space

Lemma / Def x, y, z a (geodesic) triangle in X

$\exists\, w \in [x,y]$ $u \in [y,z]$ $v \in [x,z]$ such that



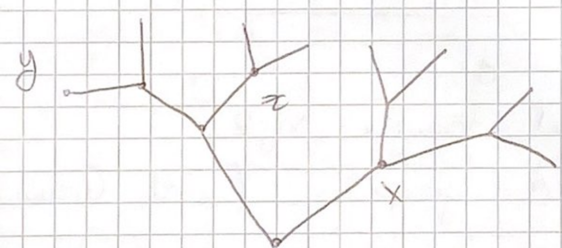
$d(x,w) = d(x,v)$
$d(y,w) = d(y,u)$
$d(z,v) = d(z,u)$



"δ-hyp" $\Leftrightarrow$ "tree-like"

$$(y|z)_x = (y,z)_x = \frac{1}{2}\left(d(x,y) + d(x,z) - d(y,z)\right)$$

$\boxed{Ex}$ in a tree



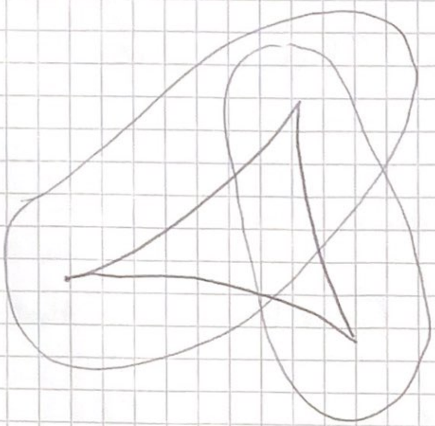
$$(y,z)_x = \frac{1}{2}(4+3-3) = 2.$$

Def X is δ-hyperbolic if for any $\Delta\, xyz$ and $y' \in [x,y]$ $z' \in [x,z]$ with $d(x,y') = d(x,z') \leq (y,z)_x$ then $d(y',z') < \delta$

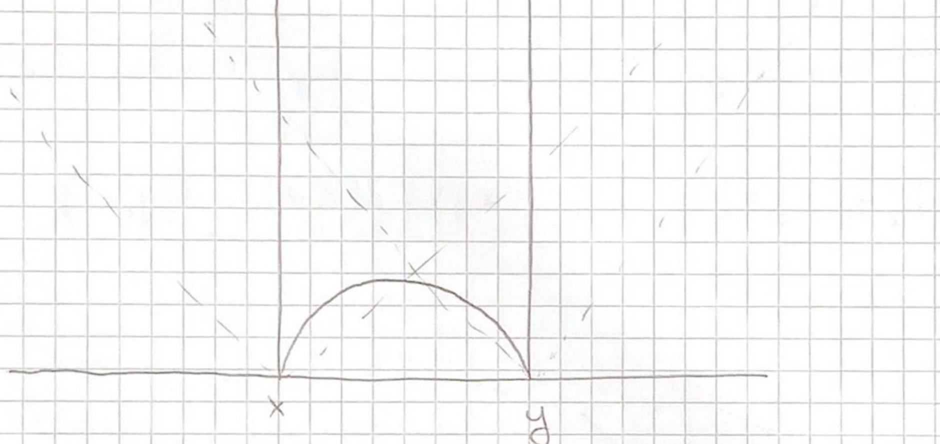
Def X is Gromov-hyperbolic if $\exists\,\delta$ such that X is δ-hyperbolic

Prop X is Gromov-hyperbolic iff $\exists\,\delta : \forall\, \Delta\, xyz$

$$[x,y] \subseteq N_\delta([x,z]) \cup N_\delta([y,z])$$



Example $\mathbb{H}^2$ is Gromov-hyperbolic



Pf idea: consider ideal triangle xyz

What about groups? A group is (word, Gromov) hyperbolic if it acts geometrically on a (proper, geodesic) hyperbolic metric space

proper discont. co-compactly, by isometries

Quasi-Isometry

$f: X \to Y$ is a QI map (quasi-isometric embedding) if $\exists\, a \geq 1 \quad b \geq 0$ such that:

$$\frac{1}{a} d(x,x') - b \leq d(f(x), f(x')) \leq a\, d(x,x') + b$$

for all $x, x' \in X$

$f : X \to Y$ is quasi-dense ($f(X)$ is a net)
if $\exists c \geq 0$ $Y \subseteq N_c(f(X))$

$f : X \to Y$ is a quasi-isometry if it is a QI embedding and $f(X)$ is a net.
(Exercise: Then $\exists g : Y \to X$ such that $f \circ g$ and $g \circ f$ are bounded distance from Id)

Stability of quasi-geodesics (Morse lemma)

If X is a δ-hyperbolic space, $c : [a,b] \to X$ (λ, ε) quasi-geodesic from p to q, $[p,q]$ is a geodesic in X

$$d_{Haus}([p,q], \operatorname{im} c) \leq R = R(\delta, \lambda, \varepsilon)$$

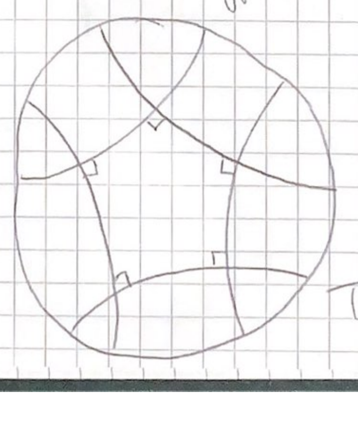
(geodesics are close)

Exercise You can use this to show: X is quasi-isometric to Y + Y is Gromov-hyperbolic $\Rightarrow$ X is Gromov-hyperbolic

If G acts geometrically on some hyperbolic space, then any space it acts on geometrically is also hyperbolic

G is hyperbolic if G is finitely generated by S and $\mathrm{Cay}(G, S)$ is hyperbolic.

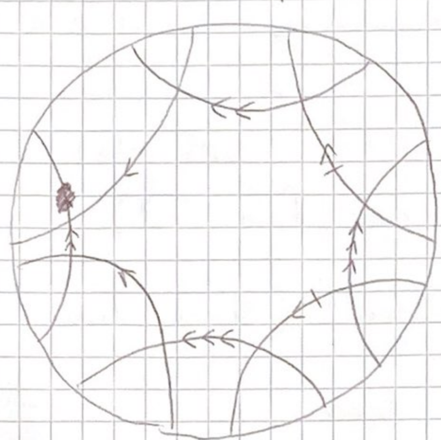
Examples



Right-angled pentagon in $\mathbb{H}^2$
Consider the gp of reflections in the sides
The images will tile $\mathbb{H}^2$

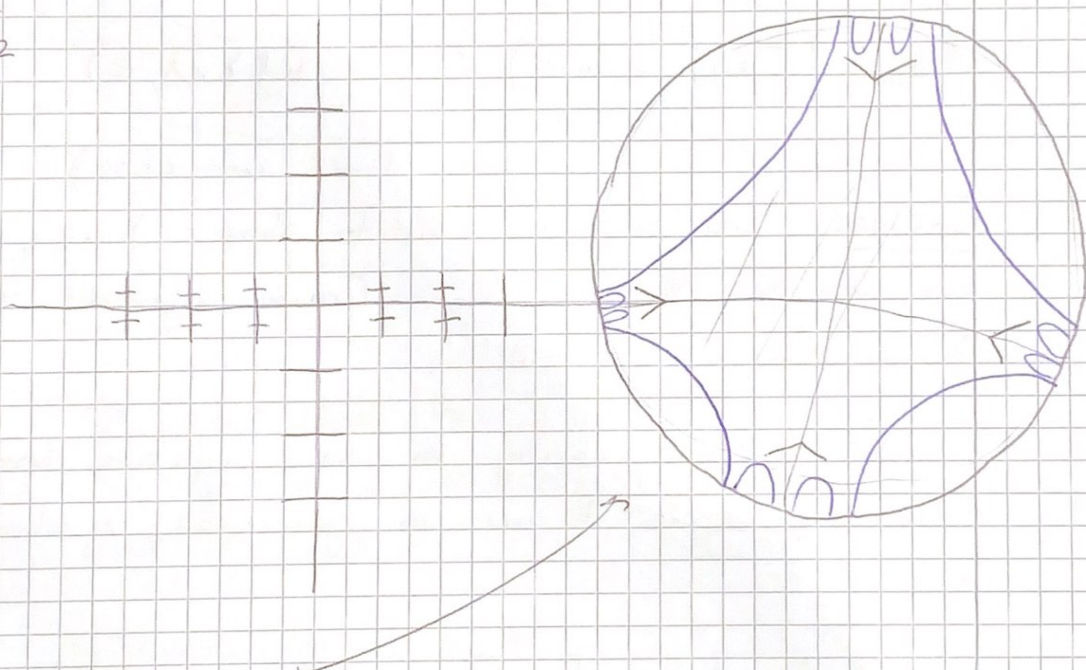
This group acts by isometries on $\mathbb{H}^2$

RACG $\langle v_i,\ v_i \in V \mid v_i v_j = v_j v_i$ if $\{i,j\} \in E,\ v_i^2 = e \rangle$

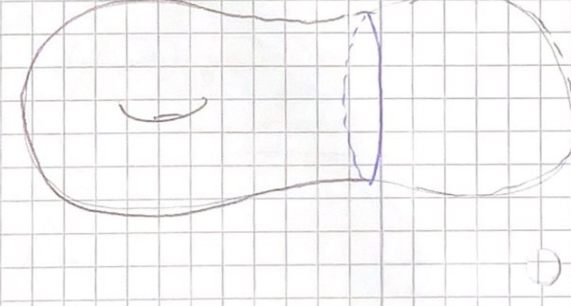


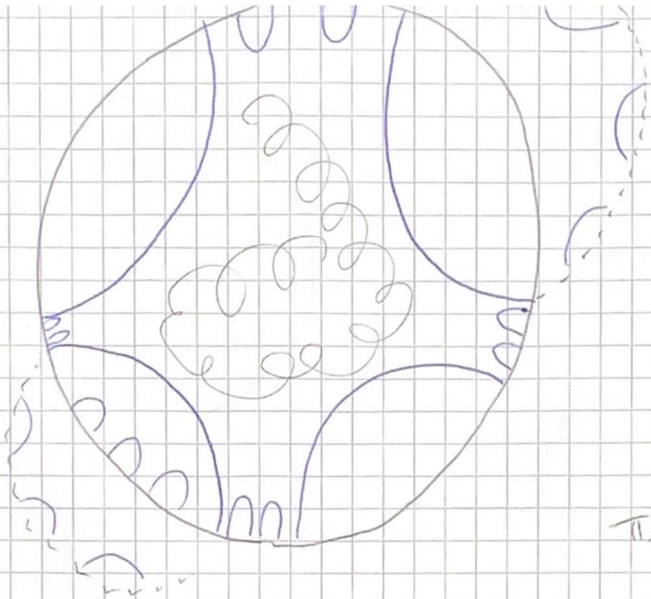
$\exists$ a hyperbolic ~~group~~ octagon w/ angles $\frac{2\pi}{8}$.

F_2

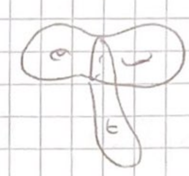


This is a convex subset of $\mathbb{H}^2$, so it is δ-hyperbolic

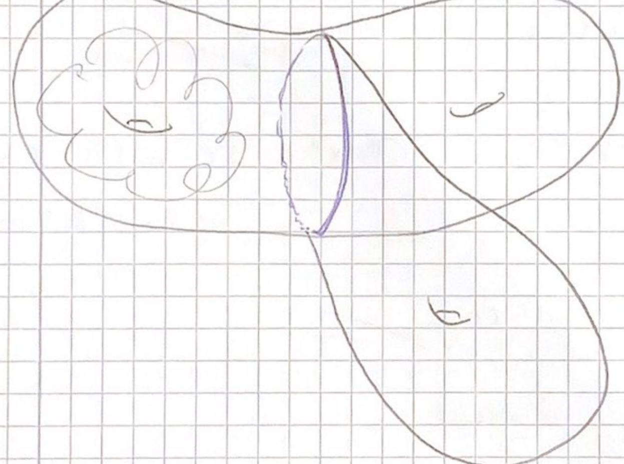




This is a hyperbolic space

π_1 ()

is a hyperbolic group

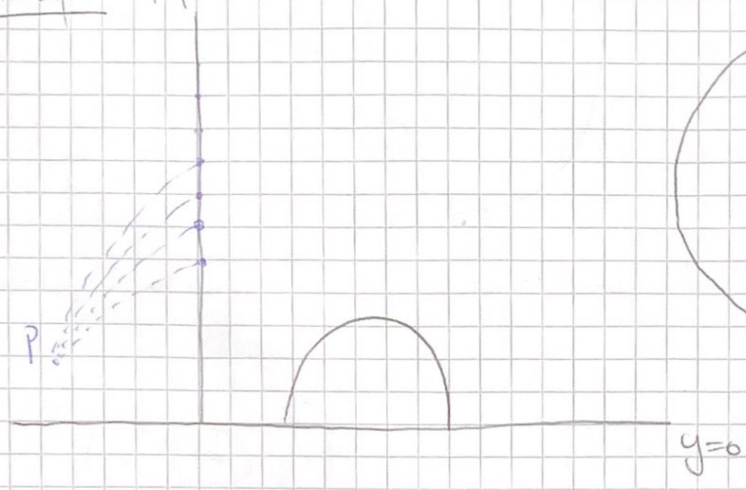


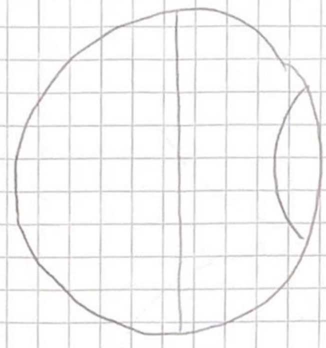
Ok what about the boundary?

X hyperbolic, ∂X topological object canonically associated to X.

- compactification of X.
- if X is quasi-isometric to Y then $\partial X \cong \partial Y$ (allows us to talk about ∂G, G hyperbolic)

Example $\mathbb{H}^2$





3 Definition X proper geodesic + hyperbolic

Def ∂X = set of geodesic rays from x_0 $/\sim$

$\sim$: they stay close together. $\gamma \sim \gamma'$ $\exists c$ $\mathrm{im}(\gamma) \subseteq N_c(\mathrm{im}\, \gamma')$
$\mathrm{im}(\gamma') \subseteq N_c(\mathrm{im}\, \gamma)$

∂X = set of quasi-geodesic rays $/\sim$

$\sim$ bounded Hausdorff distance.

Fact: in X Gromov hyperbolic space, $\exists \delta$:
$(x,y)_p \leq d(p, [x,y]) \leq (x,y)_p + \delta$

" the Gromov product approximates the distance to $[x,y]$ "

Def sequences "tending to ∞"
(x_i) sequence in X. $(x_i) \to \infty$ if $\lim\limits_{i,j \to \infty} (x_i, x_j)_p \to \infty$

(Exercise: does not depend on p)
Equivalence: $(a_i) \sim (b_i)$ if $\liminf\limits_{i \to \infty} (a_i, b_i)_p = \infty$

Rough idea why they are the same:

geodesic ray $\quad x_0 \in \{$quasi-geodesic ray$\}$

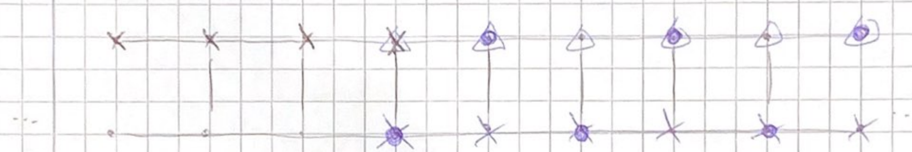
$\gamma \qquad (\gamma(i))_{i\in\mathbb{N}}$ sequence tending to ∞

quasi-geodesic ray

Topology $\qquad x_i \to m \qquad m = [x_i]$

$$(m,n)_w = \sup_{\substack{x_i \to m \\ y_i \to n}} \liminf_{i\to\infty} (x_i, y_i)_w .$$

Example $\quad \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$



$\Delta, \circ, \times, \times$ four sequences $\quad$ play with these examples !!