Minicourse: Cannon-Thurston maps

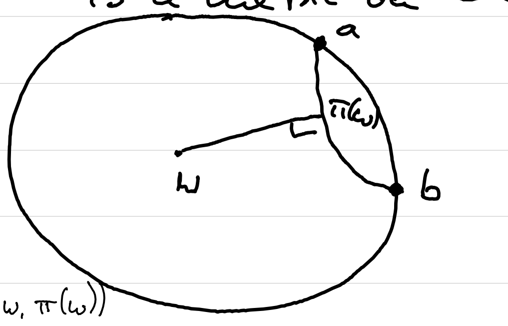
Topology on ∂X:
Boundaries mini-course

Topology:

$x_i \to m$ for m an equ. class as in III
$m = [x_i]$

$(m|n)_w = \sup_{\substack{x_i \to m \\ y_i \to n}} \liminf_{i \to \infty} (x_i|y_i)_w$

$\exists\, a > 0$: $e^{-a(m|n)_w}$ is a metric on ∂G



$d_{\partial G}(a,b) \sim e^{-d_{\mathbb{H}^2}(w,\pi(w))}$

finite

$H \leq G$: hyperbolic subgp. of hyperbolic gp.
Can assume: Generating set for G cont. gen. set for H.
Cayley-graph
$\Rightarrow$ $i: H \hookrightarrow G$ gives $i: \Gamma_H \subseteq \Gamma_G$

Q: Does $i: \Gamma_H \to \Gamma_G$ extend continuously to $\partial i: \partial H \to \partial G$?

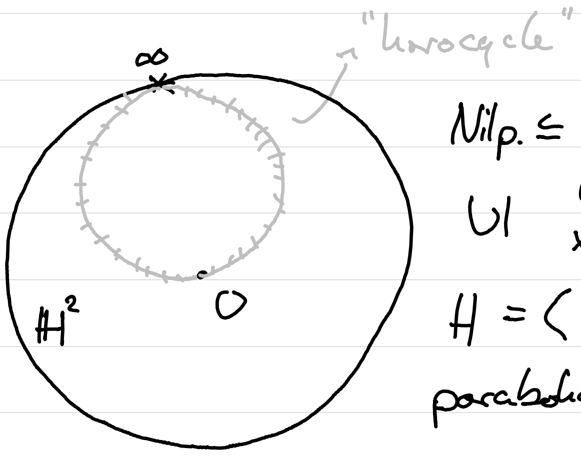
Def: Such a cts. ext., if it exists, is called a Cannon-Thurston map.

Obs: We don't need G to be a gp.
Suppose X is proper hyperbolic,
$H \curvearrowright X$ freely, prop. disc. and H hyperbolic
$\to$ gives $i: H \to X$, $h \mapsto h \cdot O$ (O a base pt. in X)
Does i extend to $\partial i: \partial H \to \partial X$?

Examples:
$H \leq G$ is a q-i emb. subgp. of a hyperb. gp. G.
Then $\partial H \hookrightarrow \partial G$ embedding

horozontal? :)

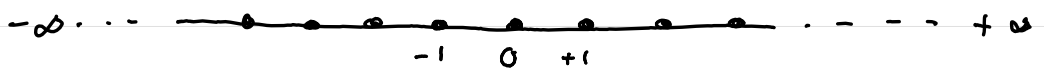
isometries:
$PSL_2(\mathbb{R})$



Nilp. $\leq PSL_2(\mathbb{R})$
$\cup \quad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $x \in \mathbb{R}$

$H = \langle \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, n \in \mathbb{Z} \rangle$
parabolic subgp.

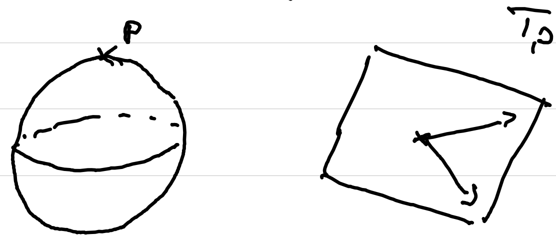
$\Gamma = \Gamma_{\mathbb{Z}}$



$\mathbb{Z} \curvearrowright \mathbb{H}^2$ via parabolic isometries does have a C-T map. (obs. needs not be inj.!)

Why bother?

Go back to Gauß curvature



"garbage?"

Principal directions

$\rightarrow$ invert the viewpoint

Look asymptotically instead of infinitesimally

What are the directions encoded by ∂H along which maximal distortion occurs?

directions in H which are most inefficient w.r.t. dist. in G.

$H \leq G$ fin. gen. subgp. of a fin. gen. gp

Compare d_H with d_G

$B_G(n)$: ball of radius n in G around 1

$H \cap B_G(n)$

$f(n) := \max\{d_H(1,h) \mid h \in B_G(n)\}$

$\hookrightarrow$ distortion function of H wrt. G.

Suppose $H \leq G$ has 1) a CT map

2) H is not q-i emb. in G (FACT)

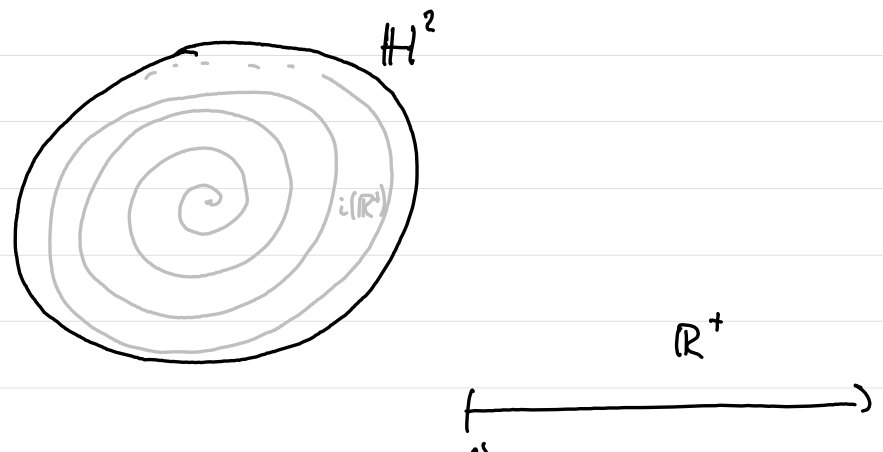
$\Rightarrow \exists\, p \neq q \in \partial H : \partial i(p) = \partial i(q)$

$\mathcal{L}_{CT} := \{ (p,q) \mid \partial i(p) = \partial i(q),\ p \neq q \in \partial H\}$

Prop.:

$\mathcal{L}_{CT} = \{\}$ iff $H \hookrightarrow G$ is q-i emb.

$\mathcal{L}_{CT}$ can be thought of as the asympt. analog of principal directions



the coll. of accumulation pts. of $i(\mathbb{R}^+) = S^1_\infty$

Answer to Q (does CT map exists for (H,G)) is NO in general (Baker-Riley '13)

Suppose $H \leq G$

$\Gamma_H \hookrightarrow \Gamma_G$ proper incl. of pts.

C-T $\iff$ proper map of pairs of pts.

$H \overset{i}{\hookrightarrow} G$ hyperbolic

h_1, h_2

$[h_1, h_2]_H$ geod. in Γ_H $\qquad$ $[i(h_1), i(h_2)]_G$ geod. in Γ_G

<u>Def.:</u> i is said to induce a <u>proper</u> map of pairs of pts. if $\exists M(N) \to \infty$ as $N \to \infty$, M prop. fn. on $\mathbb{N}$ s.t. $d_H(1, [h_1, h_2]_H) \geq N$

$$\Rightarrow d_G(1, [i(h_1), i(h_2)]_G) \geq M(N)$$

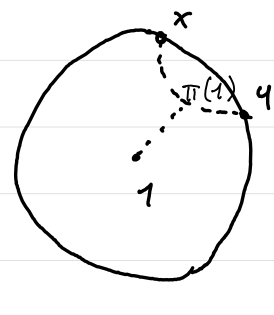
<u>Lemma:</u>
$H \overset{i}{\hookrightarrow} G$. C-T exists iff i induces a proper map of pairs of pts.

<u>Proof:</u>
Recall: $e^{-k \langle x,y \rangle_\infty^H} \sim d_{\partial H}$ visual metric

Gromov inner prod.

Continuity $\equiv$ "$d_{\partial H}(x,y)$ small $\Rightarrow d_{\partial G}(\partial i(x), \partial i(y))$ small"

Visually small sets in Γ_H go to
Visually small sets in Γ_G



$$e^{-k d_H(1, \pi(1))} \sim d_{\partial H}(x,y)$$

<u>C-T exists iff:</u> small visual diameter geodesics $[x,y]$ map to small visual diam. geodesics, i.e. if $[x,y] \subseteq \Gamma_H$ has small vis. diameter, then $[i(x), i(y)]_G$ has small vis. diameter

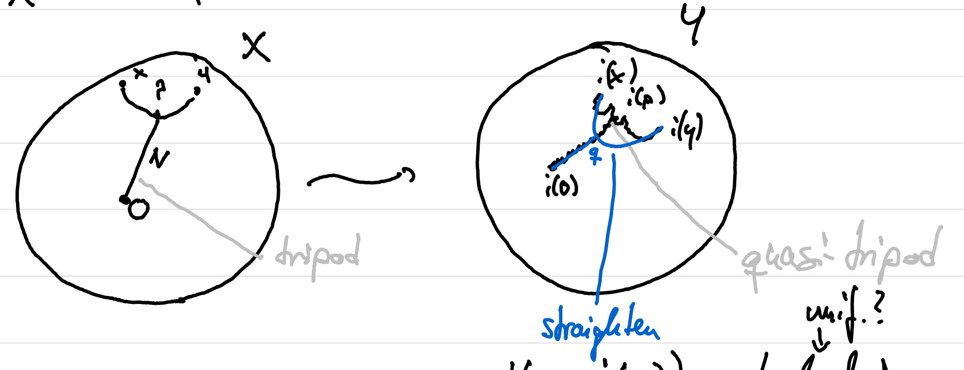
$\iff d_H(1, [x,y])$ large

$\Rightarrow d_G(1, [i(x), i(y)])$ large

$\iff$ proper embedding of pairs of pts.

QI embedding $\Rightarrow$ C-T exists

$X \hookrightarrow Y$



unif.?

$d(q, i(p))$ is bdd in terms of δ, λ, t

"quasi-tripods lie close to tripods"

If $d_H(O, [x,y]) \geq N$, then $d_G(i(O), [i(x), i(y)]) \geq M(N)$

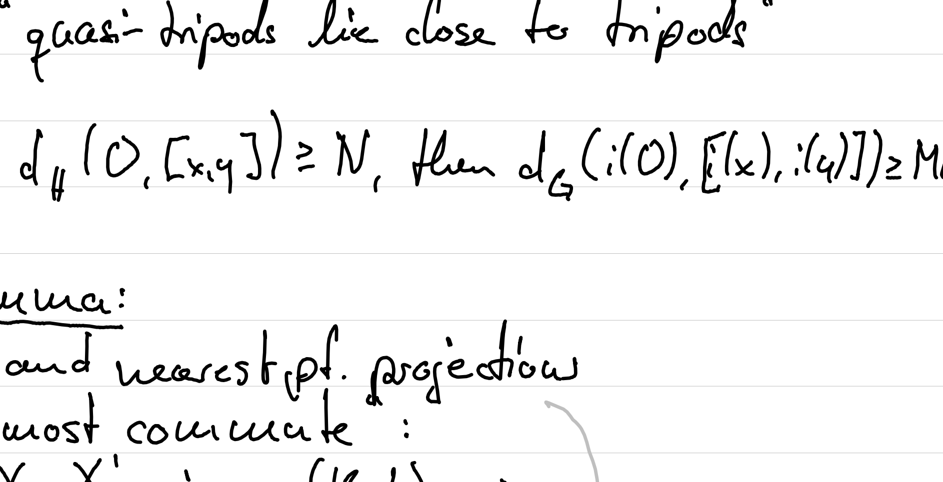
<u>Lemma:</u>
QI and nearest pt. projection "almost commute":

$f : X \to X'$ is a (K,t)-qi

$o, x, y \in X$, $\pi_X(o) \in [x,y]$ NPP

$d(f(\pi_X(o)), \pi_Y(f(o)))$ is bdd. in terms of K, t, δ

$\partial i : \partial X \to \partial Y$ is an embedding if $X \hookrightarrow Y$ is a q-i emb.



Suppose $H \hookrightarrow G$, C-T exists

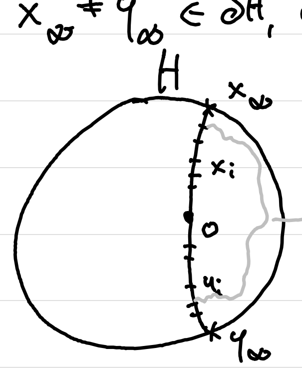
$$\mathcal{L}_{CT} = \{(p,q) \in \partial H,\ p \neq q,\ \partial i(p) = \partial i(q)\}$$

<u>Lemma:</u>
$\mathcal{L}_{CT} = \{\}$ (i.e. ∂i is an embedding)

$\Rightarrow H \hookrightarrow G$ is a qi emb.

<u>Pf.</u>
Suppose $H \hookrightarrow G$ is not a q-i emb.

$(\exists\, x_\infty \neq y_\infty \in \partial H,\ \partial i(x_\infty) = \partial i(y_\infty))$



$[i(x_n), i(y_n)]_G$ lies outside $B_m(o)$, but $[x_n, y_n]_H$ passes through o.

vis. diam.

$m \to \infty$ as $n \to \infty$

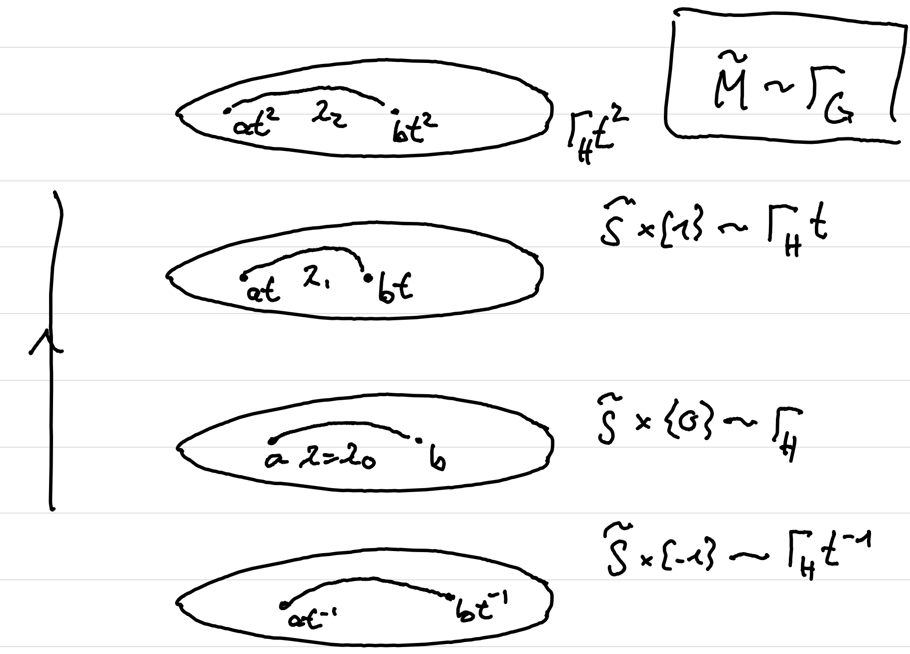
$$\mathrm{diam}_G(i(x_n), i(y_n)) \leq e^{-km}$$

$$\Rightarrow \partial i(x_\infty) = \partial i(y_\infty)$$

(3)

Thm.
$1 \to H \to G \to Q \to 1$ SES, H, G hyp.
Then, $H \hookrightarrow G$ has a CT map

Obs.
limit set $\Lambda = \partial G$
M: closed hyp. 3-mfd. fibering over S^1 w/ fibre S
$G = \pi_1(M)$
$H = \pi_1(S)$ $\qquad$ $\pi_1(S)$-equiv. map $S^1 \twoheadrightarrow S^2$
(Cannon–Thurston)



$1 \to H \to G \to \mathbb{Z} \to 1$
$\qquad\qquad\qquad\quad \overset{\shortparallel}{\langle t \rangle}$

$\mathcal{L}_\lambda = \bigcup_{i=-\infty}^{+\infty} \lambda_i$ $\quad$ | λ_i = geod. joining at^i, bt^i in $\tilde{S} \times \{i\} \sim \Gamma_H t^i = t^i \Gamma_H$

Prop: $\exists C$: $\mathcal{L}_\lambda$ is C-q.i. emb. in Γ_G.
[[Does not use hyperbolicity of G]]

$\pi_i : \tilde{S} \times \{i\} \to \lambda_i$ nearest pt. proj. of $\tilde{S} \times \{i\}$ onto λ_i w.r.t. (intrinsic) metric on $\tilde{S} \times \{i\}$
$\Pi(x) = \pi_i(x)$, if $x \in \tilde{S} \times \{i\}$

Lemma: $\exists C$: Π is a coarse C-Lipschitz retraction from Γ_G onto $\mathcal{L}_\lambda$ "thicken + ind. metric?"

Cor.: Given $\delta > 0$, $\exists D > 0$:
If G is δ-hyperbolic, then $\mathcal{L}_\lambda$ is D-quasiconvex

Obs.:
$\exists$ proper function $M(N) \xrightarrow{N \to \infty} \infty$:
$$d_H(1, \lambda) \geq N \Rightarrow d_G(1, \mathcal{L}_\lambda) \geq M(N)$$

Let μ = geod. in Γ_G joining a, b.
Then μ lies in a D-nbhd. of $\mathcal{L}_\lambda$.
Hence, (modulo Obs.), $d_H(1, \lambda) \geq N$
$$\Rightarrow d_G(1, \mu) \geq \underbrace{M(N) - D}_{\infty \text{ as } N \to \infty}$$

Hence, CT exists.

Pf. (Obs.)

$d_H(1,\lambda) = N$. Since $H \leq G$ is proper, $\exists f: \mathbb{N} \to \mathbb{N}$:

$$d_G(1,\lambda) \geq f(N)$$

$$\Rightarrow \exists K_0: \quad d_G(1,\lambda_1) \geq \max\{f(N) - K_0, 1\}$$
$$d_G(1,\lambda_m) \geq \max\{f(N) - m\cdot K_0, m\}$$

$$\Rightarrow \forall i: d_G(1, \mathcal{L}_\lambda) \geq \max\{f(N) - iK_0, i\} \geq \frac{f(N)}{K_0+1}$$

Pf. (Lemma)

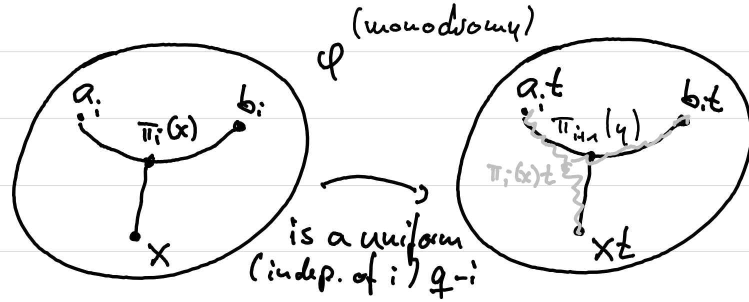
Enough to show that

$\exists C > 0$: If $x,y \in \Gamma_G$, $d_G(x,y) = 1$, then $d_G(\Pi(x), \Pi(y)) \leq C$

Case 1: $x,y \in \tilde{S} \times \{i\} = t^i \Gamma_H$, for some i

$\to$ Clear, since Γ_H is hyperbolic and $\Gamma_H t^i \approx t^i \Gamma_H$

Case 2: $y = xt$



To show: $d_G(\pi_i(x), \pi_{i+1}(y)) \leq C$

(true, since NPP's and QI's almost commute)

Observe: 1) needed a way to go up by 1 step:

$[a,b] \subseteq \Gamma_H \longrightarrow [at, bt] \subseteq \Gamma_H t$

$= t\,\Gamma_H$

2) $\forall i: \varphi_i : \tilde{S} \times \{i\} \longrightarrow \tilde{S} \times \{i+1\}$ is a (K,t)-qi

i.e. $\mathbb{Z} \to G$ $(= \pi_1(M)$ - needed a section$)$

Suffices that $Q \to G$ is a q-i section

$$1 \to H \to G \xrightarrow{\curvearrowleft \sigma} Q \to 1$$

[this is guaranteed by Mosher]

(4)

[photo]

Point-preimages
(Where is CT not 1-1?)
Equivalently, what is $\mathcal{L}_{CT}$?

Look at $\{\varphi^n(\sigma)\}_n$ – pass to a Hausdorff limit
→ gives an "almost genuine" geodesic lamination on S

Thm.:
$\mathcal{L}_{CT} = \mathcal{L}_{dynamic}$

$1 \to \pi_1(S) \to \pi_1(M) \to \mathbb{Z} \to 1$
$K \subseteq \pi_1(S)$ is f.g. inf.-index

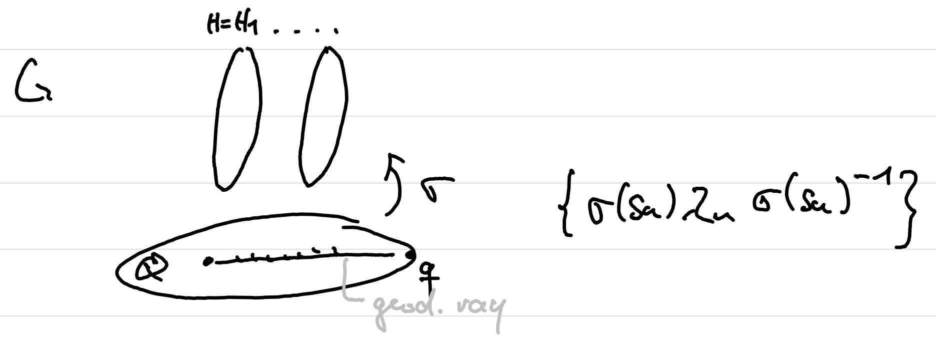
Thm. (Scott Swarup)
K is quasi-convex in $\pi_1(M)$

$K \overset{i}{\subseteq} S \overset{j}{\subseteq} M$

$(i \circ j)_* \pi_1(K) \subseteq \pi_1(M)$ has a CT map

Obs.: No leaf of $\mathcal{L}_{CT}$ is contained entirely (supported) in K.
$\Rightarrow \nexists\, l \in \mathcal{L}_{CT}(\pi_1(S), \pi_1(M))$
$\rightsquigarrow l_\pm \in \partial K \subseteq S^1_\infty = \partial(\pi_1(S))$
$\Rightarrow \mathcal{L}_{CT}(K, \pi_1(M)) = \{\}$
$\Rightarrow K$ is quasi-convex in $\pi_1(M)$

Generalisations:
1) $1 \to H \to G \to Q \to 1$ SES of hyp. gps.
Q hyp. gives ∂Q. $q \in \partial Q$ encodes a lamination $\mathcal{L}_{q,dynamic}$



Thm.:
$\mathcal{L}_{CT} = \bigcup_{q \in \partial Q} \mathcal{L}_{q,dynamic}$
For $q_1 \neq q_2$: $\mathcal{L}_{q_1,dyn} \cap \mathcal{L}_{q_2,dyn} = \{\}$
Farb-Mosher: qi rigidity
these things are mostly determined by the laminations

Generalisation of Scott-Swarup
$1 \to \pi_1(S) \to G \to Q \to 1$

Thm. (Dowdall-Kent-Leininger, Mj-Rafi)
$K \subseteq \pi_1(S)$ is fin. gen. inf. index
$\Rightarrow K$ is quasi-convex in G.

Cubulations

Q: $1 \to \pi_1(S) \to G \to \overset{Q}{\overset{\shortparallel}{F_2}} \to 1$

Assume G hyperbolic ($\Leftrightarrow$ Q is convex cocpct. in MCG)
Is G cubulable?

Partial positive answers: (Manning-Mj-Sageev)
(i.e. there are examples)

(Virt. Special)

Cubulations of hyperbolic 3-mfds. fibering over S^1
(Agol-Wise)

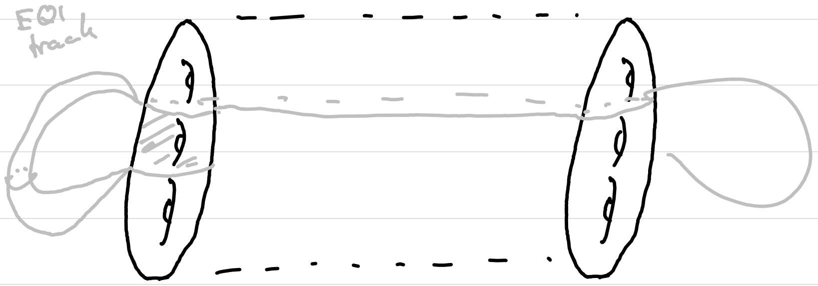
If $b_1 \geq 2$, cut along quasi-convex surface and use Wise's quasiconvex hierarchy.
If $b_1 = 1 \Rightarrow$ Need full strength of Agol's thm.

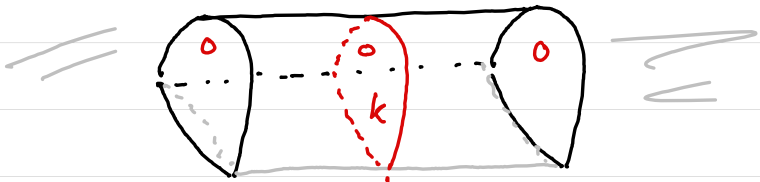
$1 \to \pi_1(S) \to \pi_1(X) \to \mathbb{Z} * \mathbb{Z} \to 1$ $\quad \pi_1(X) = G$

$K \subseteq \pi_1(S)$ is f.g. inf-index

Analog of embedded q-c incompressible surf.
EIQ track T (Dunwoody)

universal cover: $\tilde{T} \times (-1,1) \subseteq \tilde{X}$





↳ q-c by generalized Scott-Swarup
⇓
cut along these pieces to get 3-mfds. w/ bdary

Apply Wise's QC hierarchy thm. to conclude special cubulations.