Talk 4 by Mahan Mj

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Today we will try to describe the Cannon-Thurston maps.

So, to describe the Cannon-Thurston map, it suffices to describe the point-preimages :

Quesion : Where the Cannon-Thurston map is not injective ? equivalently, what is $\mathcal{L}_{CT}$? We will try to justify the name - "lamination".

$\{\Phi^n(\sigma)\}_n$: Here, let us pass to the Hausdorff limit which gives us an "almost" genuine geodesic lamination on the fibre S.

And, λ_∞ lifted to $\tilde{S} = \mathbb{H}^2$.

Here, $\mathcal{L}_{dyanamic}$ is the set of all bi-infinite geodesics and any bi-infinite geodesic corresponds to a pair of points.

Theorem : $\mathcal{L}_{CT} = \mathcal{L}_{dyanamic}$. We have $1 \to \pi_1(S) \to \pi_1(M) \to \mathbb{Z} \to 1$ and $K \subset \pi_1(S)$ is finitely generated infinite index in $\pi_1(S)$.

Theorem (Scott-Swarup) : K is quasi-convex in $\pi_1(M)$.
$K \xrightarrow{j} S \xrightarrow{i} M$ and $(i \circ j)_*(\pi_1(K)) \subset \pi_1(M)$ has a Cannon-Thurston map.

Observation : No leaf of $\mathcal{L}_{CT}$ is contained entirely(supported) in K
$\implies \exists\ l \subset \mathcal{L}_{CT}(\pi_1 S, \pi_1 M)$ such that $l_\pm \in \partial K \subset S^1_\infty = \partial\pi_1(S)$
$\implies \mathcal{L}_{CT}(K, \pi_1 M) = \varnothing \implies K$ is quasi-convex in $\pi_1 M$.

Generalizations :
(i) $1 \to H \to G \to Q \to 1$ is an exact sequence of hyperbolic groups. So, Q-hyperbolic gives ∂Q and $q \in \partial Q$ encodes a lamination $\mathcal{L}_{\{}q, dyanamic\}$ and it does generalize $\mathcal{L}_{CT} = \bigcup_{q\in\partial Q} \mathcal{L}_{\{}q, dyanamic\}$ and for $q_1 \neq q_2 \implies \mathcal{L}_{q_1} \cap \mathcal{L}_{q_2} = \varnothing$.

Generalization of Scott-Swarup theorem :

Suppose, $1 \to \pi_1 G \to G \to Q \to 1$ is given. Then we have the following :

Theorem (Dowdell-Kent-Leninger-Rafi) : $K \subset \pi_1 S$ is finitely generated infinite index $\implies K$ is quasi-convex in G.

Cubulations :

Question : Suppose, $1 \to \pi_1 S \to G \to \mathbb{F}_2 \to 1$ is given. Assume G is hyperbolic $\Longleftarrow Q$ is convex co-compact in the mapping class group. Is G "cubulable" ?
Partial results give a positive answer due to Monning, Seglen et al.

(Virtually special) Cubulation of 3-manifolds fibering over S^1(Agol-Wise) : If the 1st Betti number $b_1 \geq 2$ and then we can cut along embedded quasi-convex surface and use Wise's on quasi-convex hierarchy. If $b_1 = 1$ then we need the full strength of Agol's Theorem.

We return to $1 \to \pi_1 S \to \pi_1 M \to \mathbb{Z} * \mathbb{Z} \to 1$ where $K \subset \pi_1 S$ is a finitely generated infinite index subgroup.

Analog of enbedded quasi-convex incompressible surfaces : EIQ track T and lift everything to the universal cover : $\tilde{T} \times (-1, 1) \subset \tilde{X}$.