CANNON-THURSTON MAPS IV

Mahan Mj, Young Geometric Group Theory, 2023

Where is CTM **not** $1-1$? ← will justify language

i.e. what is the lamination $\mathcal{L}_{CT}$?

Situation: $S^1 \twoheadrightarrow S^2$

$S \hookrightarrow M^3$
$\downarrow$
S^1

How do laminations arise in the 3-manifold context?

Equip S with some hyperbolic metric. Tighten your curve to a geodesic σ. $\varphi^n \in \mathrm{Mod}(S)$ (a pseudo-Anosov) acts on σ by flowing it back towards $S \times \{0\}$.

$t^n \lambda_n t^{-n} = \lambda_0$ in $\tilde{S} \times \{t\}$ in $\tilde{M}$

Take sequence $\{\varphi^n(\sigma)\}$, pass to Hausdorff limit $\rightsquigarrow$ get an (almost) genuine geodesic lamination on S.

In 3-manifold situation,

Th. $\mathcal{L}_{CT} = \mathcal{L}_{\text{dynamic}}$ (get one for $+\infty$ unstable lamination, one for $-\infty$ stable lamination)

What does this get us?

Here $1 \to \pi_1 S \to \pi_1 M \to \mathbb{Z} \to 1$.

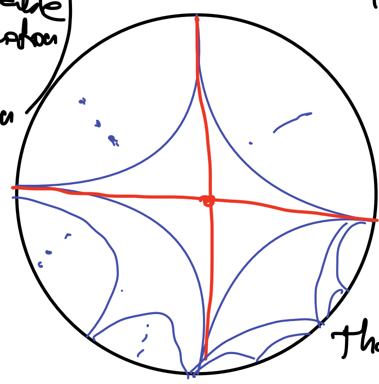
Suppose $K < S$ is f.g. and infinite index. (e.g. π_1(subsurface))

Th. (Scott-Swarup)
K is q. convex in $\pi_1(M)$.

NB: $\pi_1(S)$ is locally q. convex, so K is q. convex in $\pi_1(S)$ already.

Here $K \overset{j}{\leq} \pi_1(S) \overset{i}{\leq} \pi_1(M)$
so $(i \circ j) : \pi_1 K \leq \pi_1(M)$ has a CTM.

$\varphi_\infty(\sigma) = \lambda_\infty$ lifted to $\tilde{S} \cong \mathbb{H}^2$



This limit is $\mathcal{L}_{\text{dynamic}}$

collection of biinfinite geodesics $\longleftrightarrow$ $\mathcal{L}_{\text{dynamic}} = \left\{\begin{array}{l}\text{collection of} \\ \text{pairs of points}\end{array}\right\}$

In case of φ a pseudo-Anosov, complement of $\mathcal{L}_{\text{dynamic}}$ has area of full surface. Has trivial π_1 (as φ pA) so lifts to collection of ideal polygons.

Observation: No leaf of $\mathcal{L}_{CT}$ is contained entirely in K

Idea: no leaf is contained in a subsurface
If not π_1(subsurface), pass to a finite cover s.t. it's π_1(subsurface) by Scott's LERF theorem.
Pass to a further cover s.t. the subsurface is characteristic ... do some stuff.

That is, there is no $\ell \in \mathcal{L}_{CT}(\pi_1 S, \pi_1 M)$
s.t. $\ell_\pm \in \partial K \subseteq \partial(\pi_1(S))$

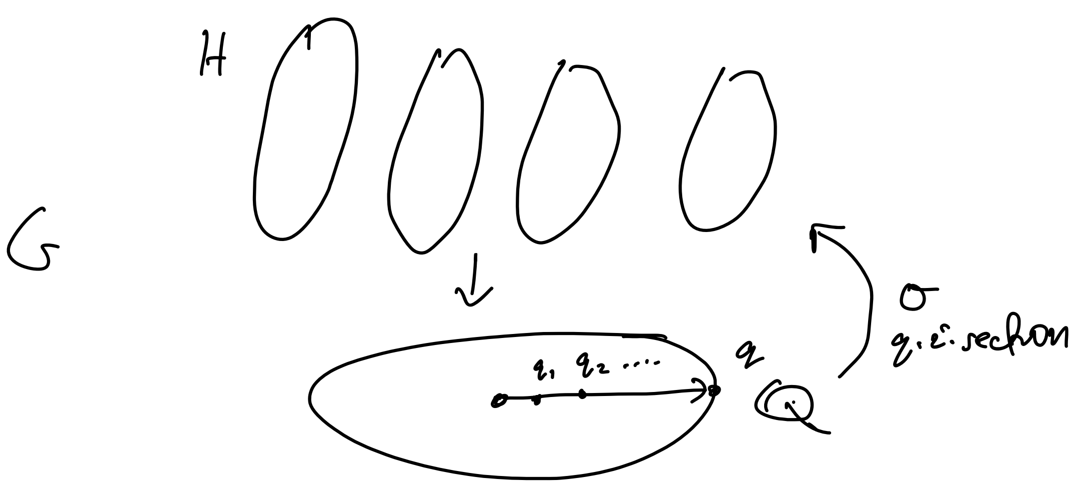
So $\mathcal{L}_{CT}(K, \pi_1 M) = \emptyset$, so K is q. convex in $\pi_1 M$.

Generalisation:

By Mosher, if H non-elementary, G, H hyperbolic,
$1 \to H \to G \to Q \to 1$ SES, then Q is hyperbolic.

Suppose all are hyperbolic.

Get ∂Q, where $q \in \partial Q$ encodes a lamination $\mathcal{L}_{q,\text{dynamic}}$.



Th. $\mathcal{L}_{CT} = \bigcup_{q \in \partial Q} \mathcal{L}_{q,\text{dynamic}}$

Play same game. Take q.i. bi-infinite q. geodesic λ_n in each sheet over q_n.
Consider
$\{\sigma(q_n)\lambda_n\sigma(q_n)^{-1}\}$

Thm. (Generalisation of Scott–Swarup)

(Dowdall–Kent–Leininger, Mj, Rafi)

Let $1 \to \pi_1 S \to G \to Q \to 1$ all hyperbolic.

If $K \leq \pi_1 S$ is f.g., infinite index, then K is q. convex in G.

More general than this. e.g. extension of F_n by fully irreducible automorphism.

Cubulations

* Q: Given $1 \to \pi_1(S) \to \underset{\pi_1(X)}{G} \to F_2 \to 1$. or $n \geq 2$

Assume G is hyperbolic (equivalently Q is convex cocompact in MCG).

Is G cubulable? (does it act geometrically on a CAT(0) cube complex?)

If so, it's special.

Partial positive answers (Manning, Mj, Sageev)
(ie. there are examples for any n.)

(v. special) Cubulation of hyperbolic 3-manifolds fibering over S^1

(Agol–Wise)

Have $b_1 \geq 1$ always as it fibres.

If $b_1 \geq 2$, have embedded q.c. surface. Cut along it, use Wise's q. convex hierarchy.

If $b_1 = 1$, need full strength of Agol's theorem.

Need embedded q. convex → as in *.

Inspiration:

If M is Haken, always have a full hierarchy.

always true if $b_1 > 0$

(can cut along incompressible surfaces to a collection of balls.)

Idea: analogue of q.c. incompressible surface:

embedded incompressible q.c. track T

Subspace w/ local product nbhd

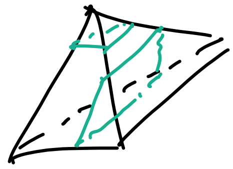
In universal cover $\tilde{T} \times (-1,1) \subseteq \tilde{X}$

$G = \pi_1(X)$

X is surface bundle over graph

T is separating.

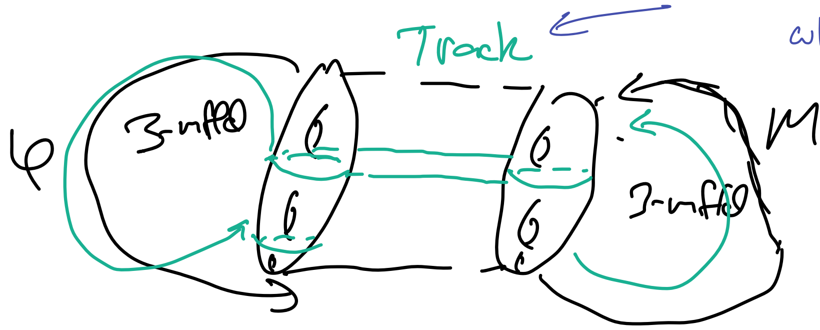
Ex. in 3-simplex: avoids all vertices

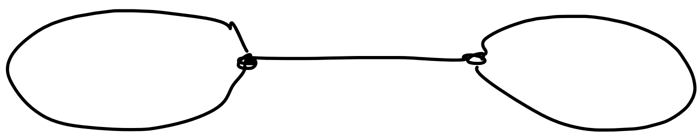
e.g. 

Idea: cutting along track is first step of hierarchy.

under some conditions, cutting along it is OK.

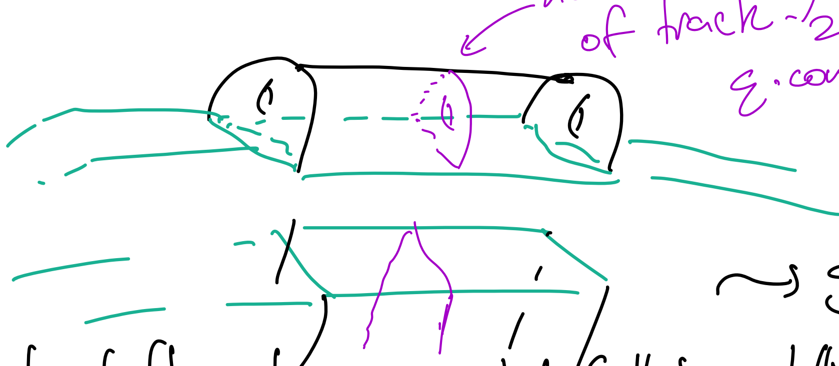
Is not q. convex in general.

Ex. 



$\pi_1 = F_2$

After cutting: get



$K = \pi_1(T_2)$ q. convex in whole H_ϕ by generalized Scott-Swarup, so is in this subgroup.

$\rightsquigarrow$ get 3-manifolds with boundary.

Need track to cut fibre into pieces which Scott-Swarup tells us are q. convex even though fibre wasn't.

Finally apply Wise's q. convex hierarchy to conclude special cubulation.