Talk 3 by Mahan Mj

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Theorem : We have an exact sequence $1 \to N \to G \to Q \to 1$, and H, G be hyperbolic and $i : H \hookrightarrow G$ has a Cannon-Thurston map.

Observation : The "Limit Set" $\Lambda_H = \partial G$.

Let's say $G = \pi_1(M)$ so M is a closed hyperbolic $3-$ manifold fibering over the circle with fibre S and $H = \pi_1(S)$ .This theorem is due to Cannon and Thurston.

The classical Cannon-Thurston map is $\pi_1 S$ equivariant from $S^1 \twoheadrightarrow S^2$.

$\mathcal{L}_\lambda = \bigcup_{i=-\infty}^{\infty} \lambda_i$ where λ_i is the geodesic joining at^i and bt^i in $\tilde{S} \times \{i\} \sim \Gamma_H t^i = t^i \Gamma_H$.

Proposition: $\exists\ C$ such that $\mathcal{L}_\lambda$ is $C-$ quasi-isometric embedded in Γ_G. [This proposition does not use hyperbolicity of G.]

$\pi_i : \tilde{S} \times \{i\} \to \lambda_i$: collection of nearest point projections of $\tilde{S} \times \{i\}$ onto λ_i with respect to metric on $\tilde{S} \times \{i\}$.

Define $\Pi(x) = \{\Pi_i(x) : x \in \tilde{S} \times \{i\}\}$.

Lemma: $\exists\ C$ such that Π is a coarse C- Lipschitz retraction from Γ_G onto $\mathcal{L}_\lambda$.

Corollary : Given $\delta > 0, \exists\ D > 0$ such that if G is $\delta-$hyperbolic then $\mathcal{L}_\lambda$ is $D-$ quasi-convex.

Observation: $\exists$ a proper function $M(N) \to \infty$ as $N \to \infty$ such that $d_H(1, \lambda) \geq N \implies d_G(1, \mathcal{L}_\lambda) \geq M(N)$ where $\lambda \subset \mathcal{L}_\lambda$. Let μ be the geodesic joining a and b. Then, μ lies in the D-neighbourhood of $\mathcal{L}_\lambda$.

Assuming the above, $d_H(1,\lambda) \geq N \implies d_G(1,\mu) \geq M(N) - D$ and $(M(N) - D) \to \infty$ as $N \to \infty$. Hence, Cannon-Thurston map exists. And, $d_H(1,\lambda) = N$

Since $H \subset G$ is proper $\implies \exists\, f : N \to \mathbb{N}$ such that $d_G(1,\lambda) \geq f(N) \implies \exists$ K_0 such that :

$d_G(1,\lambda) \geq max(f(N) - K_0, 1)$
$\implies d_G(1,\lambda_m) \geq max(f(N) - mK_0, m)$
$\implies d_G(1,\mathcal{L}_\lambda) \geq max(f(N) - iK_0, i)) \geq \frac{f(N)}{K_0+1}$ for all i.

Proof of lemma: (π-coarse Lipschitz retract) Enough to show that $\exists C > 0$ such that if $x, y \in \Gamma_G, d_G(x,y) = 1$ then $d_G(\pi(x), \pi(y)) \leq C$.

Case $1 : x, y \in \tilde{S} \times \{i\} = t^i \Gamma_H$ for some i and it's clear since we have Γ_H is hyperbolic and $\Gamma_H t^i = t^i \Gamma_H$.

Case $2 : y = xt$.

To Show That : $d_G(\pi_i(x), \pi_{i+1}(y)) \leq C$. But it's already true since (recall from last lecture) the nearest point projections and quasi-isometries almost commute.

Now,first observe that we needed a way to go up by 1 step that :

(i) $[a,b] \subset \Gamma_H \implies [at, bt] \subset \Gamma_H t = t\Gamma_H$.

(ii) $\phi_i : \tilde{S} \times \{i\} \to \tilde{S} \times \{i\}$ is a $(K,\epsilon)-$ quasi-isometry for all i i.e. $\mathbb{Z} \to G(= \pi_1 M)-$ needed a section. It suffices to show that $\sigma : Q \to G$ is a quasi-isometric section and we have the exact sequence now $1 \to H \to G \to Q \to 1$ and this splits due to the section σ from $Q \to G$ which is guaranteed by a lemma of Mosher.