Cannon-Thurston Maps III

Mahan Mj, Young Geometric Group Theory, 2023

We are now in a position to prove something.

Thm. If $1 \to H \xrightarrow{i} G \to Q \to 1$, SES

H, G hyperbolic, then $H \xhookrightarrow{i} G$ has a CTM.

generalizes the classical result.

Observation: $\Lambda_H = \partial G$ limit set

Suppose $G = \pi_1(M^3)$, M closed hyperbolic 3-manifold

fibering $\begin{matrix} S \hookrightarrow M \\ \downarrow \\ S^1 \end{matrix}$

Thm. (Cannon-Thurston)

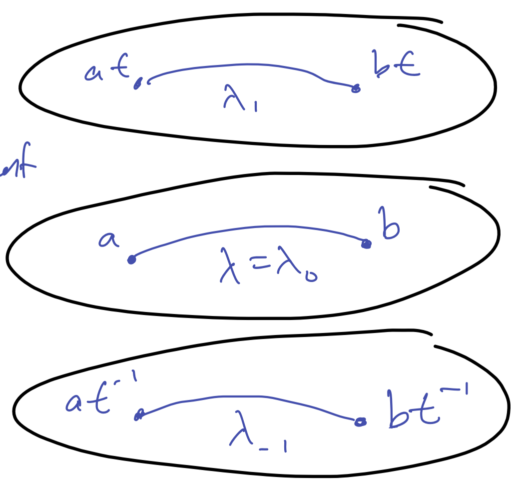
Given this setup, have a CTM on the level of universal covers $\begin{matrix} \partial \mathbb{H}^2 \to \partial \mathbb{H}^3 \\ S^1 \to S^2 \end{matrix}$

which is surjective

Consider $M^3 \overset{q.i.}{\sim} \Gamma_G$ Cayley graph of $\pi_1(M) =: G$

On the universal cover $\tilde{M}$, have $\quad \pi_1(M) \cong \pi_1(S) \rtimes \underbrace{\langle t \rangle}_{\pi_1(S^1)}$

$\tilde{M}$



geodesic segment in Γ_H

$\tilde{S} \times \{1\} \sim \Gamma_H \cdot t$

$\tilde{S} \times \{0\} \sim \Gamma_H$

$\tilde{S} \times \{-1\} \sim \Gamma_H t^{-1}$

$t = \text{gen } \pi_1(S^1)$

$H = \pi_1(S)$

$G = \pi_1(M)$

Let $\mathcal{L}_\lambda = \bigcup_{i=-\infty}^{\infty} \lambda_i$ — "ladder"

λ_i = geodesic from at^i to bt^i in $\tilde{S} \times \{i\} \sim \Gamma_H t^i = t^i \Gamma_H$ by normality.

<u>Prop.</u> There exists C s.t. $\mathcal{L}_\lambda$ is C-q.i. embedded in Γ_G.

Does <u>not</u> use hyperbolicity of G, just of H with $H \trianglelefteq G$, G f.p.

<u>Proof.</u> Let $\pi_i : \tilde{S} \times \{i\} \to \lambda_i$ is (some) nearest point projection w.r.t. intrinsic metric on $\tilde{S} \times \{i\}$. — coarsely well-defined

Let $\pi(x) = \pi_i(x)$ if $x \in \tilde{S} \times \{i\}$. (only defined on $\{\tilde{S} \times \{i\}\}$)

* <u>Lemma</u> A There exists C s.t. π is a coarse C-Lipshitz retraction from Γ_G <u>onto $\mathcal{L}_\lambda$</u>. — independent of λ → the same argument goes through for bi-infinite λ.

metric on $\mathcal{L}_\lambda$: add unif. bounded arc from $\mathcal{L}_0$ to $\mathcal{L}_1$ and so on, and take together path metric. $\mathcal{L}_\lambda$ is coarsely connected in the sense of Gromov.

<u>Cor.</u> Given $\delta > 0$, there exists $D > 0$ s.t. <u>if G is hyperbolic</u>, then $\mathcal{L}_\lambda$ is D-quasi-convex.

* <u>Observation</u> B There exists a proper function $m(N) \to \infty$ as $N \to \infty$ (independent of λ) s.t. if $d_H(1, \lambda) \geq N$, then $d_G(1, \mathcal{L}_\lambda) \geq m(N)$.

Let μ = geodesic in Γ_G, $a \overset{\mu}{\text{—}} b$

Then μ lies in a D-nbhd of $\mathcal{L}_\lambda$ (by quasi-convexity).

So (by observation) $d_H(1, \lambda) \geq N$, so

$$d_G(1, \mu) \geq m(N) - D \xrightarrow[N \to \infty]{} \infty$$

Hence CTM exists.

Need to prove *.

Aside/idea: given infinite ~~$\trianglelefteq$~~ $H \trianglelefteq G$, for any $t \in G$,

$$t\Lambda_H = \Lambda_{tH} = \Lambda_{tHt^{-1}} = \Lambda_H$$

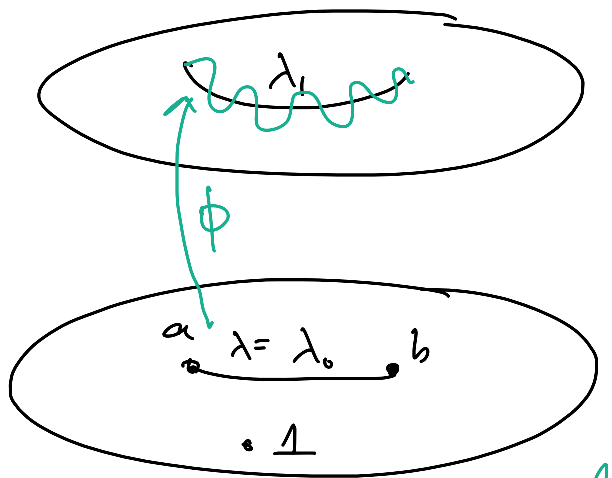
as t is bounded length

as H normal

So Λ_H is invariant under G, so $\Lambda_H = \Lambda_G = \partial G$.

(For H not normal, then, Λ_H is boundary of normaliser.)

Proof of observation B:



$d_H(1, \lambda) = N$.

Since $H \leq G$ is proper, there is $f : \mathbb{N} \to \mathbb{N}$ s.t. w/ $\lim_{n\to\infty} f(n) = \infty$

$$d_G(1, \lambda) \geq f(N)$$

So there is K_0 s.t.

$$d_G(1, \lambda_1) \geq \max(f(N) - K_0, 1)$$

so $d_G(1, \lambda_m) \geq \max(f(N) - mK_0, m)$.

$\phi(\lambda)$ is a (K, ε) q.i. where K, ε do not depend on t.

Thus $d_G(1, \mathcal{L}_\lambda) \geq \max(f(N) - iK_0, i) \geq \dfrac{f(N)}{K_0 + 1}$

which is also proper.

Idea: either horizontal or vertical coordinate is large.

Proof of Lemma A: (Π coarse Lipshitz)

Let $\Pi(x) = \Pi_i(x)$ if $x \in \tilde{S} \times \{i\}$

collection of nearest point projections.

It is enough to show that there exists $C > 0$ s.t. if $x, y \in \Gamma_G$, $d_G(x,y) = 1$, then

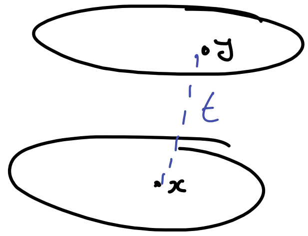
$$d_G(\Pi(x), \Pi(y)) \leq C.$$

Case 1: $x, y \in \tilde{S} \times \{i\} = t^i \Pi_H$.

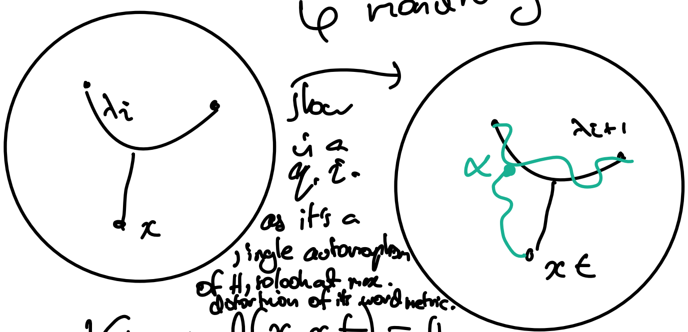
Since Π_H is hyperbolic and $\Pi_H t^i = t^i \Pi_H$,

so its nearest point projection in a δ-hyp. space.

Case 2: $y = xt$



φ monodromy



$x \mapsto t \cdot x$ is an isometry

$x \mapsto x \cdot t$ is a q. isometry

Know $d(x, xt) = 1$.

~~Want~~ Nearest point projections and QI's almost commute (from yesterday)

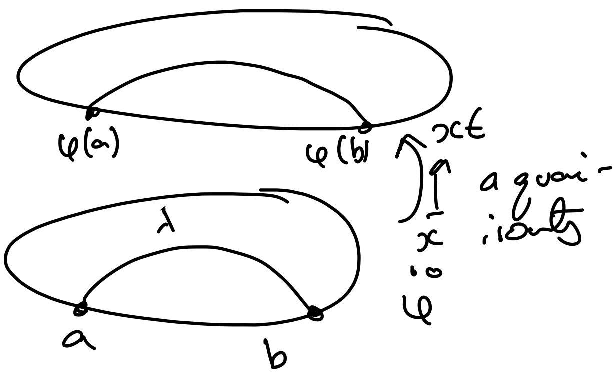
so α and the projection point in λ_{i+1} are a unif. bounded distance apart.

$$G = \coprod_{i=-\infty}^{\infty} t^i H$$

$\leftarrow \sigma(Q)$ in more general setting

Aside: biinfinite geodesics

can construct much the same picture.



φ induces $\partial\varphi : \partial H \to \partial H$

In fact, can use any q. convex set of H rather than just a geodesic.

Generalising: we need:

① a way to go up by one step

$$\varphi_i : t^i \Gamma_H \to t^{i+1} \Gamma_H$$

② φ_i is a (K,ε) q.i. $\forall i$.

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "i"] & G \arrow[r, "p"'] & Q \arrow[r] \arrow[l, bend right, "\sigma"'] & 1 \end{tikzcd}$$

③ Need $Q \to G$ a q.isometric section (before we had a splitting)

(don't need a full algebraic section)

③ Always exists by Mosher.