Talk 2 by Mahan Mj

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Today we are going to prove 3 things :

i) $i : X \to Y$ is a quasi-isometric embedding $\implies$ $\delta i : \delta X \to \delta Y$ is an embedding.

ii) $\mathcal{L}_{CT} = \varnothing \iff i$ is a quasi-isometry.

iii) CT exists $\iff$ proper embedding of pairs of points.

The 1st observation is clear from Genevieve's talk on this morning.

Let $H \subset G \implies \Gamma_H \subset \Gamma_G$ and this is a proper map and essentially it is a proper inclusion of points.

The existence of CT map is essentially saying that the map from pairs of points to pairs of points is proper. And we let $H \subset G$ be a hyperbolic subgroup.

Let, $h_1, h_2 \in H$ and let $[h_1, h_2]$ is a geodesic in Γ_H. Then $[i(h_1), i(h_2)]_G$ is a geodesic in Γ_G.

Definition : The map i is said to induce a proper map of pairs of points if $\exists$ $M(N) \to \infty$ as $N \to \infty$ such that $d_H(1, [h_1, h_2]_H) \geq N \implies d_G(1, [i(h_1), i(h_2)]_G) \geq M(N)$.

We are concerned about how the geodesics of M behaves via the proper map.

$H \subset G$ is CT $\iff$ i induces a proper map of pairs of points.

Recall: $e^{-k\langle x,y\rangle_w^H} \sim d_{\delta H}$ where the right guy has a name,"visual metric" and here continuity is equivalent to showing that $d_{\delta H}(x, y)$ small $\implies$ $d_{\delta G(\delta i(x), \delta i(y))}$ is small too. Essentially continuity is about mapping visually small sets in Γ_H to visually small sets in Γ_G.

So,$d_H(1, \pi(1)) \sim d_{\delta H}(x, y)$. The harder part is to show the existence of CT map, once done then continuity comes for free.

So CT exists iff small visual diameter geodesics $[x, y]$ map forward to small visual diameter geodesics i.e. $[x, y]_H \subset \Gamma_H$ has small visual diameter when $[i(x), i(y)]_G$ has small visual diameter $\iff d_H(1, [x, y])$ large $\implies d_G(1, [i(x), i(y)])$ is large $\iff$ proper embedding of pairs of points.

Essentially, a geodesic has a small visual diameter iff all the finite sub-segments of it has a small visual diameter.

Quasi-isometric embedding $\implies$ CT exists. So $X \hookrightarrow Y$. Now let us draw tripods and quasi-tripods.

Quasi-tripods lie close to tripods.If $d_H(o, [x, y]) \geq n \implies d_G(i(o), [i(x), i(y)]) \geq M(N)$.

Lemma: Quasi-isometries and nearest-point projections almost commute/commute coarsely i.e. if $f : X \to X^{'}$ is $o(K, \epsilon)$-quasi-isometry $\implies \pi(o) \in [x, y]$ nearest point projection and $d(f(\pi_X(o)), \pi_Y(f(o)))$ is bounded in terms of $K \in \delta$.

$\delta i : \delta X \to \delta Y$ if $X \hookrightarrow Y$ is a quasi-isometric embedding.

So, point 3 and point 1 has been taken care of.

Suppose,$H \subset G$ and CT exists. Then, define $\mathcal{L}_{CT} := \{(p, q) \in \delta H, p \neq q; \delta i(p) \neq \delta i(q)\}$.

Lemma : $\mathcal{L}_{CT} = \varnothing$ (i.e. δ is an embedding) $\implies H \subset G$ is a quasi-embedding.

Rough Proof: Suppose $H \subset G$ is not a qi embedding. Then $\exists\, x_\infty \neq y_\infty \in \delta H, \delta i(x_\infty) = \delta i(y_\infty)$. Suppose $[i(x_n), i(y_n)]_G$ lies outside $B_m(o)$ but $[x_n, y_n]_H$ passes via o. Given any m, we choose x_n, y_n such that the geodesic with them as end-points passes via o. ($m \to \infty$ as $n \to \infty$) . $Diameter_G(i(x_n), i(y_n)) \leq e^{-km} \implies \delta i(x_\infty) = \delta i(y_\infty)$.