CANNON-THURSTON MAPS II

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Suppose $H \leq G$. Have $\Gamma_H \hookrightarrow G$

proper inclusion of points

Idea of Cannon Thurston map: proper inclusion of pairs of points.

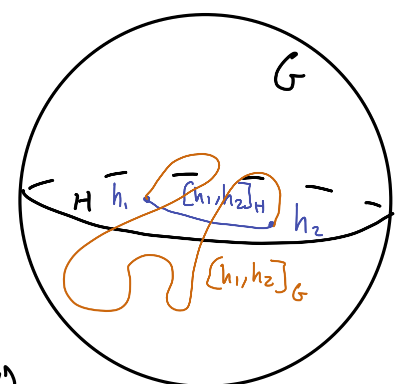
Setup:

$H \overset{i}{\subseteq} G$ hyperbolic, $h_1, h_2 \in H$.

Have $[h_1, h_2]_H$ geodesic in H, and $[i(h_1), i(h_2)]_G$ geodesic in G.

Defn. i induces a proper map of pair of points if there exists $M(N)$ s.t. $\lim_{N \to \infty} M(N) = \infty$, i.e. M is a proper function of N
s.t. if $d_H(1, [h_1, h_2]) \geq N$ then $d_G(1, [i(h_1), i(h_2)]) \geq M(N)$.

Lemma $H \leq G$ has a CTM iff i induces a proper embedding of pairs of points.

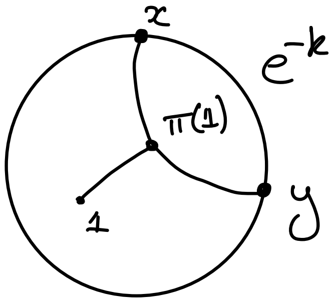


Proof. Recall $d_{\partial H} \sim e^{-k\langle x,y\rangle_w}$ visual metric

within a multiplicative constant of

w.r.t. H

Idea: The map $\partial i : \partial H \to \partial G$ is continuous, so visually small sets in Γ_H go to visually small sets in Γ_G.



$e^{-k d_H(1,\pi(1))} \sim d_{\partial H}(x,y)$ so

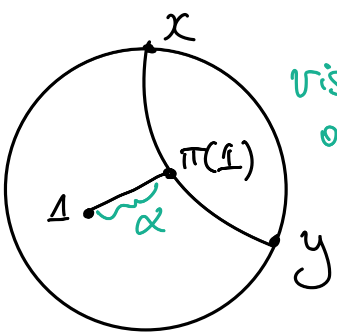
Instead of talking about points on boundary, talk about (biinfinite) geodesics.

CTM exists iff small visual diameter geodesics $[x,y]$ map forward to small vis. diam. (SVD) geodesics. *A biinfinite geodesic has SVD iff all finite subsets of it have SVD. So replace $[x,y]$ with arbitrary finite subset.

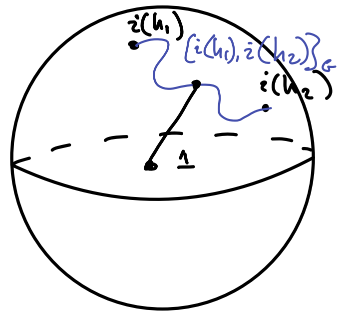
That is, if $[x,y]_H$ has SVD then $[i(x),i(y)]_G$ has SVD.
Now in H, not ∂H.

Equivalently, $d_H(1,[x,y])$ large means $d_G(1,[i(x),i(y)])$ large, which is true iff you have proper embedding of pairs of points.

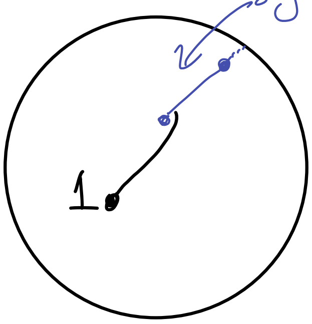
Properness $\iff$ SVD sets go to SVD sets $\iff$ CTM is well-defined?



vis. diam. of $[x,y] \sim e^{-\alpha}$



Well-definedness:



segment of infinite ray is uniformly small visual diameter

so will map to uniformly SVD in G, so will get well-defined image in the boundary.

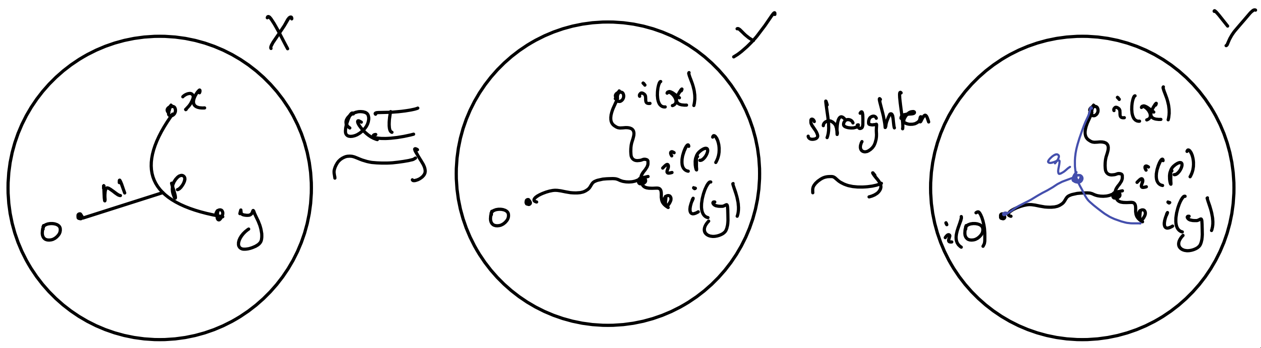
the spiral map $\mathbb{R}^+ \hookrightarrow \mathbb{H}^2$ is not a proper map of points. Can get

$d_{\mathbb{R}}(m,n) > N \Rightarrow d_{\mathbb{H}^2}(m,n) \geq M(N)$

but not $d_{\mathbb{H}^2}(1,[m,n])$.

QI-embedding implies CTM exists

Suppose $X \overset{QI}{\hookrightarrow} Y$.



Tripod

quasi-tripod

Use Morse Lemma to give that $i([x,y])$ is close to $[i(x), i(y)]$ so $d(q,p)$ is bounded in terms of δ, QI constants.

If $d_X(0,[x,y]) \geq N$, then
$d_Y(i(0), [i(x, i(y))]) \geq M(N)$ for $M(N)$ some linear function.

So CTM exists and furthermore you get that it is an embedding.

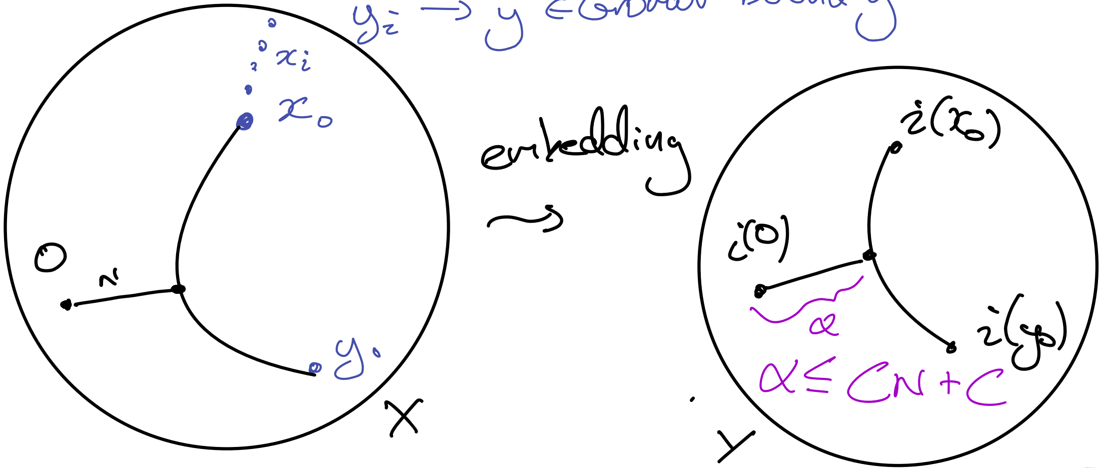
Lemma. QI's and nearest point projections almost commute.

Proof. $f: X \to X'$ is $O(K, \epsilon)$ - q.i.
For $o, x, y \in X$, $\pi_X(o) \in [x,y]$ NPP
Get $d(f(\pi_X(o)), \pi_Y(f(o)))$ bounded by K, ϵ, λ.

NB: not true in CAT(0) spaces. e.g. $\mathbb{R}^2$.

Lemma. $\partial i : \partial X \to \partial Y$ is an embedding if $X \hookrightarrow Y$ is a QI-embedding.

$x_i \longrightarrow \bar{x} \in$ Gromov boundary
$y_i \longrightarrow \bar{y} \in$ Gromov boundary



$\alpha \leq CN+C$ so distance of $\bar{x}, \bar{y}$ in ∂Y is bounded below

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Goal: $\mathcal{L}_{CT} = \emptyset \iff$ QI

Suppose $H < G$ and CTM exists.

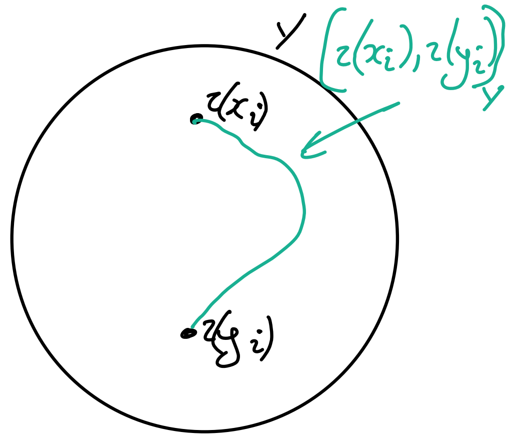
$$\mathcal{L}_{CT} = \{(p,q) \in \partial H,\ p \neq q,\ \partial i(p) = \partial i(q)\}$$

Lemma. Suppose not q.i. embedding. Then $\mathcal{L}_{CT}$ not empty.

Proof.

I'm not sure exactly how you cook up there x_i, y_i s.t. x_i converges and y_i converges to point in ∂X.

For every M,
have x_M, y_M s.t.
$[z(x_M), z(y_M)]_Y$ lies outside
$B_M(0) \subset X$, but $[x_M, y_M]_H$
passes through 0.



So $[\bar{x}, \bar{y})$ is SUD in Y but not in X.

So $\cancel{x}\ \partial z(\bar{x}) = \partial z(\bar{y})$ but $\bar{x} = \bar{y}$.