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CANNON-THURSTON MAPS :

About the Topology : $x_i \to m$ and we write $m = [x_i]$ with $(m,n)_w = \sup_{x_i \to m, y_i \to n} \liminf_{i \to \infty} (x, y_i)_w$ and an example of $\mathbb{Z} \times \mathbb{Z}/2$. One can play with these examples.

So, what is the topology on δX if $\exists\ a > 0$ such that $e^{-a(m,n)_w}$ is a metric space on δG. We think of $d_{\delta G}(a,b) \sim e^{-d_{\mathbb{H}^2}(w,\pi(w))}$.

$H \subset G$ is a hyperbolic subgroup of a hyperbolic group. We can always assume that finite generating set for G that contains generating set for $H \implies i : H \hookrightarrow G$ gives $i : \Gamma_H \subset \Gamma_G$ where Γ_G is the Cayley graph of G.

Question : Does $i : \Gamma_H \hookrightarrow \Gamma_G$ extend continuously to $\delta i : \delta H \to \delta G$?

Definition : Such a continuous extension,if it exists is called a **CANNON-THURSTON** map.

When does a Cannon-Thurston map exists? : We usually don't need G to be a group.

Suppose : X is proper-hyperbolic and H acts freely and properly discontinuously, and then H is a hyperbolic group gives $i : H \to X$ by $h \mapsto h.0$ (let 0 is a fixed base-point).

Does i extend to $\delta i : \delta H \to \delta X$?

Examples: Let, $H \subset G$ is a quasi-isometrically embedded subgroup of a hyperbolic group G and then $\delta H \hookrightarrow \delta G$ embedding and the isometric group of the Hyperbolic plane is given by $\mathrm{PSL}_2(\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1\}/\{\pm Id\}$ and let $\mathcal{N} \subset \mathrm{PSL}_2(\mathbb{R})$ be a nilpotent subgroup and note $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ with $x \in \mathbb{R}$. Let, $H = \langle \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \rangle$ where $n \in \mathbb{Z}$, then H is a Parabolic subgroup. Let us draw a horocycle. So $\mathbb{Z}$ acting on the hyperbolic plane via parabolic isometries do have a Cannon-Thurston map.

The point of this example is that the Cannon-Thurston map **need not** be injective.

Why do we at all bother ? We go back to Gaussian curvature and think about the principle directions.

Let us invert our view-point : Look asymptotically instead of infinitesimally.

We now ask "What are the directions encoded by δH along which maximal distortion occurs ?"

Here one way of thinking is that those are precisely the directions in $\mathbb{H}$ which are most inefficient with respect to distance in G So $H \subset G$ is a finitely generated subgroup of a finitely generated group and then we compare distance d_H with d_G.

We write $B_G(n) :=$ Ball of radius n in G. we look at $H \cap B_G(n)$ and $f(n) = max(d_H(1, h) : h \in B_G(n) \cap H)$ and where $f(n)$ is the distortion function of H wrt G.

Digression: The linearity of distortion function is analogous to totally geodesicness of submanifolds.

Suppose (i) $H \subset G$ has a Cannon-Thurston map, (ii) H is not quasi-isometrically embedded in $G \implies \exists\, p \neq q \in \delta H$ such that $\delta i(p) = \delta i(q)$.

Define $\mathcal{L}_{CT} := \{(p, q) | \delta i(p) \neq \delta i(q), p \neq q \in \delta H\}$

Proposition : Suppose (H, G) has a Cannon-Thurston map. Then, $\mathcal{L}_{CT} = \varnothing$ iff $H \subset G$ is quasi-isometrically embedded. $\mathcal{L}_{CT}$ can be thought of as the asymptotic analog of principle directions.

An example where Cannon-Thurston map does not exist : Collection of accumulation points of $i(\mathbb{R}^+) = S^{'}_{\infty}$.

An answer to the question of existence of Cannon-thurston map for (H, G) is "No" in general (Baker-Riley '13) .