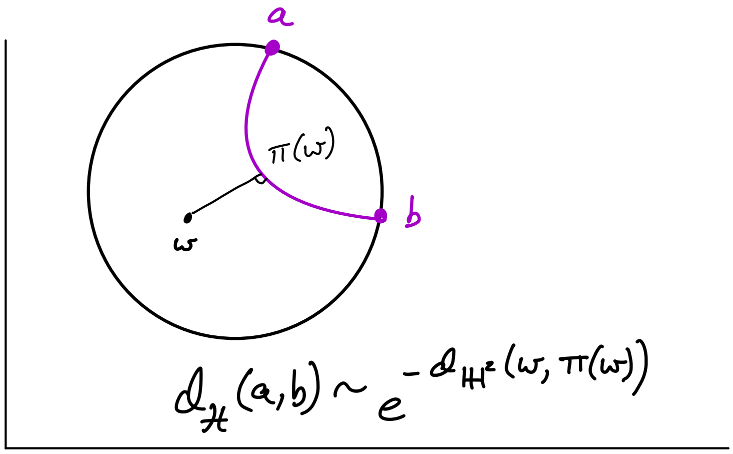
Cannon-Thurston Maps I

Mahan Mj, Young Geometric Group Theory, 2023

Topology on ∂X: $\exists a > 0$ s.t. $e^{-a(m,n)_w}$ is a metric on ∂X.



$H \leq G$ hyperbolic subgroup of a hyperbolic group.

Fact. hyp. groups are finitely presented, so can assume generating set for G contains a gen. set for H. So have $i : \Gamma_H \hookrightarrow \Gamma_G$ ← Cayley graph of H

Question: Does $i : \Gamma_H \hookrightarrow \Gamma_G$ extend continuously to the boundary?

Defn. Such an extension, if it exists, is called a Cannon-Thurston map. (CTM)

Observation G doesn't need to be a group.

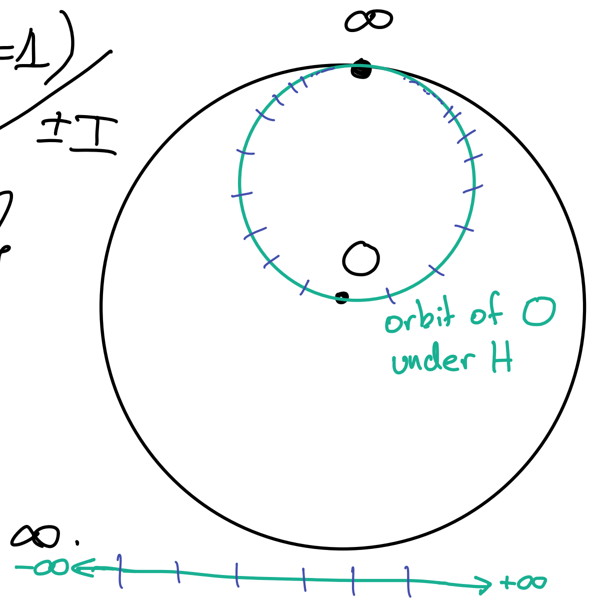
Setup: X proper hyperbolic, $H \curvearrowright X$ (freely) properly discontinuously, H hyperbolic. Get $i : H \to X$

$h \mapsto h(p)$ ← basepoint

Does i extend to $\partial i : \partial H \to \partial X$?

Ex. (1) $H \subset G$ q.i.-embedded subgroup of a hyp. group G. Then CTM exists and $\partial H \to \partial G$ is an embedding.

② $PSL_2\mathbb{R} = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \text{determinant} = 1 \right) \Big/ \pm I$



$\text{Nilp} \subseteq PSL_2\mathbb{R} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$

$\cup$

$H = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\}$

H is "parabolic subgroup". Fixes ∞.

$H \bullet O$ gives Cayley graph of $\mathbb{Z}$ in $\mathbb{H}^2$, embedded in a horocycle. $\Gamma_{\mathbb{Z}}$

$\partial \Gamma_{\mathbb{Z}} = \{-\infty, +\infty\}$.

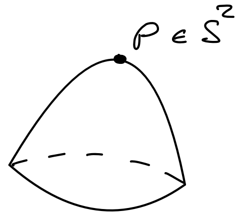
$\mathbb{Z} \curvearrowright \mathbb{H}^2$ via parabolic isometries does have a CTM. But not injective.

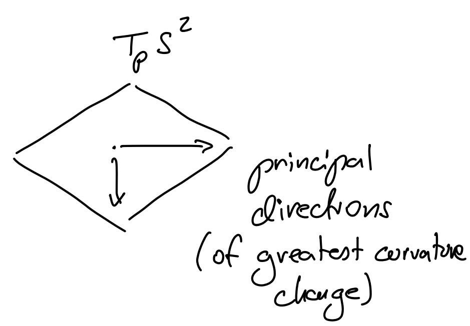
$\partial i : \partial \mathbb{Z} \to \partial \mathbb{H}^2 \cong S^1$

point $\mapsto$ fixed point of parabolic group

MOTIVATION

Gaussian curvature:





Coarse geometry: Look asymptotically instead of infinitesimally.

CTMs encode these directions of maximal distortion.

What are the directions encoded by ∂H along which maximal distortion occurs?

i.e. directions in H that are most inefficient w.r.t. distance in G.

Let $H \subseteq G$ a f.g. subgroup of a f.g. group.
Assume gen. set of G contains one of H.
Compare d_H with d_G

$B_G(n) :=$ ball of radius n in G about $1 \in G$.
in Cayley graph Γ_G

$f(n) := \max\left(d_H(1,h) \mid h \in H \cap B_G(n)\right)$

distortion function of H w.r.t. G

Exact value depends on chosen generating set, but form of function is a QI-invariant.

A submanifold N of M is totally geodesic if for $p,q \in N$, $d_N(p,q) = d_M(p,q)$. Distortion function gives us an analogue of this $\rightsquigarrow$ its being QI-embedded.

What about principal curvature?
Suppose ① $H \subseteq G$ has a CTM. (So H, G both hyperbolic)
② H is not QI-embedded in G

Consequence: have $p \neq q \in \partial H$ s.t. $\partial i(p) = \partial i(q)$.

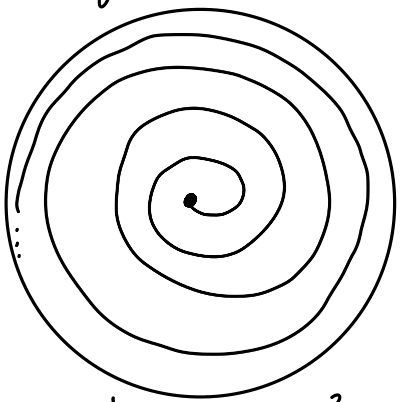
Let $\mathcal{L}_{CT} = \{(p,q) \mid \partial i(p) = \partial i(q),\ p \neq q \in \partial H\}$

Cannon-Thurston lamination

Prop. Suppose $H \hookrightarrow G$ has a CTM. NB: CTM is uniquely defined if it exists.
$\mathcal{L}_{CT} = \emptyset$ iff $H \hookrightarrow G$ is QI-embedded

Rmk. $\mathcal{L}_{CT}$ can be thought of as the coarse analogue of principal directions.

Examples when CTM does not exist:

① 

$\mathbb{R}^+ \hookrightarrow \mathbb{H}^2$
(subspace not subgroup)

$\mathbb{R}^+ \bullet \longrightarrow +\infty$

Set of accumulation points of

$i(\mathbb{R}^+) = S^1_\infty$ circle at infinity

So CTM does not exist as it would not be well-defined

② $H \leq G$, hyp. subgroup of hyp. group, CTM does not always exist. (Baker–Riley '13, using techniques from small cancellation theory)

But generally when CTM exists is not well-understood.
The obstruction to its existence is well-defined-ness —
you get continuity for free.